Operads, Duality and the Gravity Operad

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Introduction

First introduced by May in [May72], operads are a crucial part of modern mathematics. Roughly speaking, an operad is a sequence $(\mathscr{P}(n))_{n\in\mathbb{N}}$ where $\mathscr{P}(n)$ corresponds to a collection of n-ary operations. These are equipped with certain structure morphisms corresponding to inserting the result of an operation into another one, and these structure maps are subject to certain relations dictated by the fact that a different ordering of multiple insertions may yield the same result. Thus, beyond their initial application in homotopy theory, operads provide a general framework for dealing with "structures with (higher) operations".

The main parts of this thesis assume some familiarity with basics of operads. We refer to [LV12] for a contemporary depiction of various aspects of them.

As monoid objects in an abelian category with a monoidal structure, operads can be considered as generalizations of rings and thus algebraic objects on their own. In fact, one of the important breakthroughs in the theory of operads was the generalization of concepts such as quadratic algebras, their Manin products and Koszul duality to the context of operads as initiated by Ginzburg and Kapranov in [GK94]. This thesis deals with two phenomena that are significant examples of this aspect of operads.

The first section of this thesis is dedicated to considering quadratic algebras and ultimately quadratic operads as an instance of a general concept of duality. This concept, which Boyarchenko and Drinfeld study in [BD13] under the name of "Grothendieck–Verdier categories", deals with monoidal categories $(\mathbf{M}, \otimes, \mathbb{1})$ together with a "dualizing object" $K \in \mathbf{M}$ such that for every $Y \in \mathbf{M}$, the functor $X \mapsto \mathrm{Hom}(X \otimes Y, K)$ is representable by an object $D(Y) \in \mathbf{M}$ and such that these objects can be assembled into an equivalence $D\colon \mathbf{M}^{\mathrm{op}} \to \mathbf{M}$ (cf. Definition 1.4). It is an abstract formulation of Verdier duality and other similar phenomena in algebraic geometry.

As pointed out by Manin in [Man17], quadratic algebras and quadratic operads yield examples of Grothendieck–Verdier categories where the monoidal structure is given by the respective black product and the duality functor D is given by the respective quadratic duality (cf. Remark 1.17 and Corollary 1.45). However, it turns out that these examples unfortunately do not satisfy many properties which are prominent in other examples (cf. [BD13, 1.2]). For example, the dualizing object is not the monoidal unit (cf. Proposition 1.22 resp. Proposition 1.48). Moreover, for quadratic operads, one cannot even construct a comparison morphism $X \otimes Y \to D^{-1}(D(X) \otimes D(Y))$ (cf. Proposition 1.49).

The other example of operadic algebra which this thesis deals with is the Koszul duality between the hypercommutative operad and the gravity operad. The hypercommutative operad is given by the homology of compactified moduli spaces of marked genus 0 curves and has been important in mathematical physics since Kontsevich and Manin showed in [KM94] that the quantum cohomology of a projective variety (over $\mathbb{Q}$) yields a hypercommutative algebra. The gravity operad, as shown by Ginzburg and Kapranov in [GK94] and by Getzler in [Get95], is the Koszul dual of the hypercommutative operad and related to the moduli spaces of smooth marked genus 0 curves. This can be seen as an analogue of the Koszul duality between the commutative and the Lie operad – in fact, the former is a suboperad of the hypercommutative operad and the latter is a suboperad of the gravity operad.

Based on the approach of [GK94] and [Get95], the second section of this thesis gives a detailed description of the gravity operad (cf. Subsection 2.1) and concludes with its relation to the hypercommutative operad via Koszul duality (cf. Theorem 2.7). This approach to the gravity operad uses some geometric properties of moduli spaces of marked genus 0 curves extensively. Therefore, the required results about these moduli spaces are reviewed in Appendix B. Moreover, Appendix A deals with basics on trees, which are crucial in dealing with operads and stratifications of the above mentioned compactified moduli spaces.

Notations and Conventions

Here we fix some notations and conventions which are used throughout the thesis.

Notation 0.1. When dealing with functors and natural transformations, we usually use the symbols ▲, ■, ⬟ and ⬣ as variables which can stand for both objects and morphisms. For example, the Hom-functor could be denoted by Hom(■, ⬟).

For monoidal categories, we use standard terminology which can be found, for example, in [Mac98] or [BL11].

Notation 0.2. When we say that $(\mathbf{M}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ is a monoidal category we mean the following:

- $\mathbf{M}$ is the underlying category,
- $\otimes\colon \mathbf{M} \times \mathbf{M} \to \mathbf{M}$ is the monoidal product,
- $\mathbb{1} \in \mathbf{M}$ is the unit object,
- α: (■ ⊗ ⬟) ⊗ ⬣ ⇒ ■ ⊗ (⬟ ⊗ ⬣) is the associator,
- λ: $\mathbb{1}$ ⊗ ■ ⇒ ■ is the left unitor,
- ρ: ■ ⊗ $\mathbb{1}$ ⇒ ■ is the right unitor.

We may suppress parts of this data when they are clear from the context.

Notation 0.3. When dealing with braided monoidal categories we will denote the braiding ■ ⊗ ⬟ ⇒ ⬟ ⊗ ■ with τ.

We also need to fix notation for some concepts from basic (linear) algebra.

Notation 0.4. We denote by Σ_n the symmetric group on n letters.

Convention 0.5.
- For a set S, let $\mathbb{C}^S$ denote the free $\mathbb{C}$-vector space on S. When working with graded vector spaces, we put each basis element of S into degree 1, so that $\mathbb{C}^S$ is concentrated in degree 1.
- For a finite dimensional (graded) $\mathbb{C}$-vector space V let $\det(V)$ denote its determinant, i. e. its highest exterior power $\bigwedge^{\dim V} V$. Note that if V is concentrated in degree k, then $\det(V)$ is concentrated in degree $k \cdot \dim(V)$.

Remark 0.6. The functor

$$\begin{aligned} \mathrm{FinSet} &\to \mathrm{GrVect}_{\mathbb{C}} \\ S &\mapsto \det(\mathbb{C}^S) \end{aligned}$$

from the category of finite sets to the category of graded $\mathbb{C}$-vector spaces can be made into a strong monoidal functor $(\mathrm{FinSet}, \amalg, \varnothing) \to (\mathrm{GrVect}_{\mathbb{C}}, \otimes_{\mathbb{C}}, \mathbb{C})$. Indeed, we have an isomorphism $\bigwedge^0 \mathbb{C}^{\varnothing} \cong \mathbb{C}$ and for finite sets S, S' there is a natural isomorphism

$$\begin{aligned} \det(\mathbb{C}^S) \otimes \det(\mathbb{C}^{S'}) &\xrightarrow{\cong} \det(\mathbb{C}^{S \amalg S'}) \\ (v_1 \wedge \ldots \wedge v_{|S|}) \otimes (v'_1 \wedge \ldots \wedge v'_{|S'|}) &\mapsto (v_1 \wedge \ldots \wedge v_{|S|} \wedge v'_1 \wedge \ldots \wedge v'_{|S'|}). \end{aligned}$$

We we will often have a decomposition $S = S' \dot\cup S''$ of a finite set as a disjoint union and consider the induced isomorphism $\det(\mathbb{C}^{S'}) \otimes \det(\mathbb{C}^{S''}) \cong \det(\mathbb{C}^S)$ on determinants. Such isomorphisms will be denoted by ϕ with some decoration.

Next, we fix some conventions about operads. As mentioned before, we will not review the general theory of operads, but instead use [LV12] as a reference.

Convention 0.7.
- By a Σ-module in a category $\mathbf{C}$ we mean collection $E = (E(n))_{n \in \mathbb{N}_+}$ of objects in $\mathbf{C}$ such that each $E(n)$ is equipped with an action of Σ_n. Morphisms of Σ-modules are defined to be levelwise equivariant morphisms.
- By a (co)operad we mean a non-unital symmetric (co)operad. While dealing with concrete (co)operads, we will use the definition given by infinitesimal (co)compositions. Sometimes we will consider elements of the n-th level of an operad as n-ary operations. We refer to [LV12, 5.3] for a various definitions of operads and to [LV12, 5.7] for cooperads.
- For a Σ-module E, by $\mathfrak{F}(E)$ we denote the free operad generated by E (as in [LV12, 5.4] and [GK94, 2.1]).
- For an operad $\mathcal{P}$, we denote by $\Omega\mathcal{P}$ its cobar construction (as in [LV12, 6.5] and [GK94, 2.1]).

We will also need some notation about (co)homology of spaces.

Notation 0.8. Let X, Y be spaces.

- When no explicit coefficients are given, $H_\bullet(X)$ resp. $H^\bullet(Y)$ will denote homology resp. cohomology with $\mathbb{C}$-coefficients.
- The Künneth isomorphisms $H_\bullet(X \times Y) \cong H_\bullet(X) \otimes H_\bullet(Y)$ and $H^\bullet(X \times Y) \cong H^\bullet(X) \otimes H^\bullet(Y)$ (in $\mathbb{C}$-coefficients) as well as variations of these will be denoted by κ.

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1 Grothendieck–Verdier Categories and Manin Products

In this section we interpret Manin products of quadratic algebras (as defined in [Man87]) resp. quadratic operads (as defined in [GK95]) in the context of Grothendieck–Verdier categories, i. e. monoidal categories with a "dualizing object".

The moral of our considerations can be summarized as follows: Both for quadratic algebras and quadratic operads the respective white product can be obtained by transferring the respective black product along the respective quadratic duality, which yields examples of Grothendieck–Verdier categories. However, neither of these examples are r-categories. For quadratic algebras, one can still construct a comparison morphism between the two monoidal products in question even though these do not form an r-category.

We mostly follow the conventions of Boyarchenko and Drinfeld from [BD13], but variants of the concept that they call "Grothendieck–Verdier categories" was studied by Barr under the name of "$*$-autonomous categories" in [Bar79] and later works. It was observed in [Man17] that quadratic algebras and quadratic operads fit into the framework of Grothendieck–Verdier categories and we elaborate on the ideas presented there.

1.1 The Setup

We start with some generalities about transferring monoidal structures along equivalences.

Notation 1.1. For the rest of this subsection we fix a monoidal category $(\mathbf{M}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ and an equivalence $D\colon \mathbf{M} \to \mathbf{M}^{\mathrm{op}}$. Further, we fix a quasi-inverse $D^{-1}\colon \mathbf{M}^{\mathrm{op}} \to \mathbf{M}$ along with natural isomorphisms $\eta\colon D^{-1}D \cong \mathrm{id}_{\mathbf{M}}$ and $\varepsilon\colon DD^{-1} \cong \mathrm{id}_{\mathbf{M}^{\mathrm{op}}}$.

Remark 1.2. Transferring the monoidal structure on $\mathbf{M}$ and its opposite along D, we obtain new monoidal structures on $\mathbf{M}$:

$(\mathbf{M}, \ominus, D^{-1}(\mathbb{1}), \bar{\alpha}, \bar{\lambda}, \bar{\rho})$ is given by the product

$$\blacksquare \ominus \pentagon = D^{-1}(D(\blacksquare) \otimes D(\pentagon)),$$

where the associator $\bar{\alpha}$ is the composition

$$\begin{aligned}
& D^{-1}(D(D^{-1}(D(\blacksquare) \otimes D(\pentagon))) \otimes D(\hexagon)) \\
\xRightarrow{D^{-1}(\varepsilon \otimes \mathrm{id})} \quad & D^{-1}((D(\blacksquare) \otimes D(\pentagon)) \otimes D(\hexagon)) \\
\xRightarrow{D^{-1}(\alpha^{-1})} \quad & D^{-1}(D(\blacksquare) \otimes (D(\pentagon) \otimes D(\hexagon))) \\
\xRightarrow{D^{-1}(\mathrm{id} \otimes \varepsilon^{-1})} \quad & D^{-1}(D(\blacksquare) \otimes D(D^{-1}(D(\pentagon) \otimes D(\hexagon)))),
\end{aligned}$$

the left unitor $\bar{\lambda}$ is given by

$$\begin{aligned}
& D^{-1}(D(D^{-1}(\mathbb{1})) \otimes D(\blacksquare)) \\
\xRightarrow{D^{-1}(\varepsilon \otimes \mathrm{id})} \quad & D^{-1}(\mathbb{1} \otimes D(\blacksquare)) \\
\xRightarrow{D^{-1}(\lambda^{-1})} \quad & D^{-1}(D(\blacksquare)) \\
\xRightarrow{\eta} \quad & \blacksquare,
\end{aligned}$$

and the right unitor $\bar{\rho}$ by

$$\begin{aligned}
& D^{-1}(D(\blacksquare) \otimes D(D^{-1}(\mathbb{1}))) \\
\xRightarrow{D^{-1}(\mathrm{id} \otimes \varepsilon)} \quad & D^{-1}(D(\blacksquare) \otimes \mathbb{1}) \\
\xRightarrow{D^{-1}(\rho^{-1})} \quad & D^{-1}(D(\blacksquare)) \\
\xRightarrow{\eta} \quad & \blacksquare.
\end{aligned}$$

Similarly, $(\mathbf{M}, \odot, D^{-1}(\mathbb{1}), \dot{\alpha}, \dot{\lambda}, \dot{\rho})$ is given by

$$\blacksquare \odot \pentagofill = D^{-1}(D(\pentagofill) \otimes D(\blacksquare))$$

and the "reversed" associator resp. unitors. In other words, $(\mathbf{M}, \odot, D^{-1}(\mathbb{1}))$ is the opposite of the monoidal structure $(\mathbf{M}, \ominus, D^{-1}(\mathbb{1}))$.

In [BD13] the authors work with $\odot$, but we will work with $\ominus$ which fits better to the classical definition of black and white products of quadratic algebras resp. operads.

Next, we recall some definitions from [BD13].

Definition 1.3. An object $K \in \mathbf{M}$ is called *dualizing* if for every $Y \in \mathbf{M}$ the functor $\mathrm{Hom}(\blacksquare \otimes Y, K)$ is representable by an object $D(Y) \in \mathbf{M}$ (in which case these objects can be assembled into a functor $D\colon \mathbf{M} \to \mathbf{M}^{\mathrm{op}}$) and $D\colon \mathbf{M} \to \mathbf{M}^{\mathrm{op}}$ is an equivalence.

Definition 1.4. A *Grothendieck–Verdier category* (or *GV category*) is a monoidal category together with a dualizing object.

Definition 1.5. A Grothendieck–Verdier category is called an *r-category* if the chosen dualizing object is the monoidal unit. (In particular, in this case the monoidal unit is a dualizing object.)

If one also assumes that the Grothendieck–Verdier category in question is an r-category, one can easily construct a natural transformation $\blacksquare \otimes \pentagofill \Rightarrow \blacksquare \odot \pentagofill$ which is compatible with the respective associators. However, in the cases of quadratic algebras resp. operads with the black product, which are of interest for us, the monoidal unit is not a dualizing object (cf. Proposition 1.22 resp. Proposition 1.48). In the case of quadratic algebras a morphism between the two products can still be constructed even though quadratic algebras with quadratic duality do not form an r-category (cf. Proposition 1.23).

1.2 Quadratic Algebras

We start by establishing some conventions and recalling some definitions.

Notation 1.6. We fix a ground field k for the rest of this section. All algebras, vector spaces etc. are going to be over k.

The operation $\blacksquare^*$ will denote k-duality, i. e. the functor $\mathrm{Hom}_k(\blacksquare, k)$. For a subspace W of a k-vector space V, we will denote by $W^\perp$ the subspace $\{f \in V^* \mid \forall w \in W : f(w) = 0\} \subseteq V^*$.

Moreover, $\otimes$ will denote the tensor product over k. While working with such concrete monoidal structures, we will be less pedantic about the associators and unitors.

Definition 1.7.
- A *quadratic algebra* $A = \bigoplus_{n \in \mathbb{N}} A_n$ is an $\mathbb{N}$-graded algebra which is a quotient of a free algebra $\mathrm{F}(V)$ on a finite dimensional $\mathbb{N}$-graded vector space which is concentrated in degree 1 by an ideal generated by a space of relations $R \subseteq V \otimes V = \mathrm{F}(V)_2$.
- The category $\mathbf{QA}$ of quadratic algebras has as objects quadratic algebras and as morphisms algebra morphisms which respect the grading.

Notation 1.8. Given a quadratic algebra A, A_1 is its space of generators. We will denote its space of relations by $\mathrm{R}(A) \subseteq A_1 \otimes A_1$.

Given a finite dimensional vector space V and a subspace $R \subseteq V \otimes V$, $\mathrm{A}(V, R)$ will denote the quadratic algebra $\mathrm{F}(V)/(R)$.

Remark 1.9. Given quadratic algebras A and B, restriction to the first graded piece induces a bijection

$$\mathrm{Hom}_{\mathbf{QA}}(A, B) \cong \{f \in \mathrm{Hom}_k(A_1, B_1) \mid (f \otimes f)(\mathrm{R}(A)) \subseteq \mathrm{R}(B)\}$$

by the universal property of free algebras and quotient algebras.

Definition 1.10. The *quadratic dual* construction on $\mathbf{QA}$ is given by

$$\blacksquare^! = \mathrm{A}(\blacksquare_1^*, \mathrm{R}(\blacksquare)^\perp)$$

where we implicitly identify $\blacksquare_1^* \otimes \blacksquare_1^*$ with $(\blacksquare_1 \otimes \blacksquare_1)^*$.

Remark 1.11. The quadratic dual construction defines an equivalence $\mathbf{QA} \to \mathbf{QA}^{\mathrm{op}}$. Indeed, identifying the double dual of a finite dimensional vector space with the space itself and the “double complement” of a subspace with itself, we see that $((\blacksquare)^!)^! \cong \blacksquare$.

As we are going to see in Corollary 1.19, this equivalence is in fact induced by a dualizing object with respect to a monoidal structure on $\mathbf{QA}$.

We now recall the definitions of black and white products of quadratic algebras following [Man87].

Notation 1.12. Let $\sigma_{2,3}$ denote the natural transformation

$$\blacktriangle \otimes \blacksquare \otimes \text{⬟} \otimes \text{⬢} \Rightarrow \blacktriangle \otimes \text{⬟} \otimes \blacksquare \otimes \text{⬢}$$

which swaps the second and the third component.

Definition 1.13. The *black product* of quadratic algebras is defined as

$$\blacksquare \bullet \text{⬟} = \mathrm{A}(\blacksquare_1 \otimes \text{⬟}_1, \sigma_{2,3}(\mathrm{R}(\blacksquare) \otimes \mathrm{R}(\text{⬟}))).$$

Remark 1.14. Let V be a finite-dimensional vector space, $R \subseteq V \otimes V$ a subspace. Consider the natural isomorphism $\rho\colon V \otimes k \cong V$ given by the right unitor of the monoidal structure on k-vector spaces. Then, $\mathrm{F}(\rho)_2\colon (V \otimes k) \otimes (V \otimes k) \cong V \otimes V$ maps $\sigma_{2,3}(R \otimes k \otimes k)$ isomorphically to R. Thus, by Remark 1.9, we have a natural isomorphism $\mathrm{A}(V, R) \bullet \mathrm{A}(k, k \otimes k) \cong \mathrm{A}(V, R)$ of quadratic algebras.

In fact, also the left unitor, the associator and the braiding of $(\mathrm{Vect}_k, \otimes_k, k)$ extend in a similar way to quadratic algebras to yield a symmetric monoidal structure on **QA** with $\bullet$ as the monoidal product and $\mathrm{A}(k, k \otimes k)$ as the monoidal unit.

Definition 1.15. The *white product* of quadratic algebras is defined as

$$\blacksquare \circ \text{⬟} = \mathrm{A}(\blacksquare_1 \otimes \text{⬟}_1, \sigma_{2,3}(\mathrm{R}(\blacksquare) \otimes \text{⬟}_1^{\otimes 2} + \blacksquare_1^{\otimes 2} \otimes \mathrm{R}(\text{⬟}))).$$

Remark 1.16. Let V be a finite-dimensional vector space, $R \subseteq V \otimes V$ a subspace. Consider the natural isomorphism $\rho\colon V \otimes k \cong V$ given by the right unitor of the monoidal structure on k-vector spaces. Then, $\mathrm{F}(\rho)_2\colon (V \otimes k) \otimes (V \otimes k) \cong V \otimes V$ maps $\sigma_{2,3}(R \otimes k \otimes k) = \sigma_{2,3}(R \otimes k \otimes k + V \otimes V \otimes 0)$ isomorphically to R. Thus, by Remark 1.9, we have a natural isomorphism $\mathrm{A}(V, R) \circ \mathrm{A}(k, 0) \cong \mathrm{A}(V, R)$ of quadratic algebras.

In fact, also the left unitor, the associator and the braiding of $(\mathrm{Vect}_k, \otimes_k, k)$ extend in a similar way to quadratic algebras to yield a symmetric monoidal structure on **QA** with $\circ$ as the monoidal product and $\mathrm{A}(k, 0)$ as the monoidal unit.

Remark 1.17. The monoidal structure of Remark 1.16 is essentially the one obtained by transferring the black product along the quadratic dual construction (as in Remark 1.2). Indeed, given two quadratic algebras A and B, the generating space of $(A^! \bullet B^!)^!$ is

$$(A_1^* \otimes B_1^*)^* \cong ((A_1 \otimes B_1)^*)^* \cong A_1 \otimes B_1 = (A \circ B)_1$$

and its space of relations is given by

$$\begin{aligned}\sigma_{2,3}(\mathrm{R}(A)^\perp \otimes \mathrm{R}(B)^\perp)^\perp &\cong \sigma_{2,3}((\mathrm{R}(A)^\perp)^\perp \otimes ((B_1^*)^*)^{\otimes 2} + ((A_1^*)^*)^{\otimes 2} \otimes (\mathrm{R}(B)^\perp)^\perp) \\ &\cong \sigma_{2,3}(\mathrm{R}(A) \otimes B_1^{\otimes 2} + A_1^{\otimes 2} \otimes \mathrm{R}(B)) = \mathrm{R}(A \circ B).\end{aligned}$$

In order to put quadratic algebras into the context of GV categories, we will need the following adjunction.

Remark 1.18. Let V, W, X be finite-dimensional vector spaces and $R \subseteq V \otimes V$, $S \subseteq W \otimes W$ resp. $T \subseteq X \otimes X$ subspaces. Consider the natural isomorphism

$$\mathrm{Hom}_k(V \otimes W, X) \cong \mathrm{Hom}_k(V, W^* \otimes X)\,. \tag{1.1}$$

For $f \in \mathrm{Hom}_k(V \otimes W, X)$ and its adjoint $\hat{f} \in \mathrm{Hom}_k(V, W^* \otimes X)$ we have $\mathrm{F}(f)_2(\sigma_{2,3}(R \otimes S)) \subseteq T$ if and only if $\mathrm{F}(\hat{f})_2(R) \subseteq \sigma_{2,3}(S^\perp \otimes X^{\otimes 2} + (W^*)^{\otimes 2} \otimes T)$.

Thus, by Remark 1.9, the isomorphism of (1.1) extends to a natural adjunction isomorphism

$$\mathrm{Hom}_{\mathbf{QA}}(\mathrm{A}(V,R) \bullet \mathrm{A}(W,S), \mathrm{A}(X,T)) \cong \mathrm{Hom}_{\mathbf{QA}}\Big(\mathrm{A}(V,R), \mathrm{A}(W,S)^! \circ \mathrm{A}(X,T)\Big)\,.$$

Now that we have the required tools about quadratic algebras at hand, we can interpret these in the context of GV categories as it was remarked in [Man17].

Corollary 1.19. *The unit $\mathrm{A}(k,0)$ of the white product of quadratic algebras is a dualizing object for the black product of quadratic algebras since for all $A \in \mathbf{QA}$, the functor $\mathrm{Hom}_{\mathbf{QA}}(\blacksquare \bullet A, \mathrm{A}(k,0)) \cong \mathrm{Hom}_{\mathbf{QA}}\big(\blacksquare, A^! \circ \mathrm{A}(k,0)\big)$ is representable by $A^! \circ \mathrm{A}(k,0) \cong A^!$.*

Next, we want to show that $\mathbf{QA}$ together with the black product is not an r-category.

Definition 1.20.
- We call a quadratic algebra A a *square-zero extension* if $\mathrm{R}(A) = A_1 \otimes A_1$.
- Given a finite-dimensional vector space V, we denote by $\mathrm{SqZ}(V)$ the square-zero extension $\mathrm{A}(V, V \otimes V)$.

Remark 1.21. If A is a square-zero extension and B a quadratic algebra, then $\mathrm{Hom}_{\mathbf{QA}}(B, A) \cong \mathrm{Hom}_k(B_1, A_1)$ via restriction to the first graded piece.

Indeed, every algebra morphism $f\colon B \to A$ is uniquely determined by its restriction $f_1\colon B_1 \to A_1$ by Remark 1.9 and the condition $(f_1 \otimes f_1)(\mathrm{R}(B)) \subseteq \mathrm{R}(A)$ is trivially fulfilled if $\mathrm{R}(A) = A_1 \otimes A_1$, so every linear map $B_1 \to A_1$ extends to an algebra homomorphism $B \to A$.

Proposition 1.22. *The monoidal unit $\mathrm{A}(k, k \otimes k)$ of the black product of quadratic algebras is not a dualizing object, i. e. there is no equivalence $D\colon \mathbf{QA} \to \mathbf{QA}^{\mathrm{op}}$ s. t. for all $B \in \mathbf{QA}$ the functor $\mathrm{Hom}_{\mathbf{QA}}(\blacksquare \bullet B, \mathrm{A}(k, k \otimes k))$ is represented by $D(B)$.*

Proof. Note that $\mathrm{A}(k, k \otimes k)$ is a square-zero extension. Hence, by Remark 1.21, for $A, B \in \mathbf{QA}$ we have isomorphisms

$$\begin{aligned}\operatorname{Hom}_{\mathbf{QA}}(\blacksquare \bullet B, \mathrm{A}(k, k \otimes k)) &\cong \operatorname{Hom}_k((\blacksquare \bullet B)_1, \mathrm{A}(k, k \otimes k)_1) \\ &= \operatorname{Hom}_k(\blacksquare_1 \otimes B_1, k) \\ &\cong \operatorname{Hom}_k(\blacksquare_1, B_1^*) \\ &\cong \operatorname{Hom}_{\mathbf{QA}}(\blacksquare, \operatorname{SqZ}(B_1^*)) \,.\end{aligned}$$

Thus, if $\mathrm{A}(k, k \otimes k)$ were a dualizing object, $D(B)$ would be isomorphic to $\operatorname{SqZ}(B_1^*)$ by the uniqueness of representing objects. However, a functor with such values on objects cannot be an essentially surjective since (for instance) the quadratic algebra $\mathrm{A}(k, 0)$ is not in its essential image because it is not a square-zero extension. □

Even though the monoidal unit of $\bullet$ is not a dualizing object, we can still obtain a natural transformation from $\bullet$ to $\circ$.

Proposition 1.23. *Given quadratic algebras A and B, the identity morphism $A_1 \otimes B_1 \to A_1 \otimes B_1$ extends to a natural morphism $A \bullet B \to A \circ B$ which is compatible with the respective associators.*

Proof. Since

$$\sigma_{2,3}(\mathrm{R}(\blacksquare) \otimes \mathrm{R}(\blacklozenge)) \subseteq \sigma_{2,3}(\mathrm{R}(\blacksquare) \otimes \blacklozenge_1^{\otimes 2} + \blacksquare_1^{\otimes 2} \otimes \mathrm{R}(\blacklozenge)),$$

the identity map $A_1 \otimes B_1 \to A_1 \otimes B_1$ extends to a morphism $A \bullet B \to A \circ B$ of quadratic algebras by Remark 1.9.

Since the associator of $\bullet$ resp. $\circ$ is an extension of the associator of $\otimes$ on the level of generators, the compatibility with the associators follows from the compatibility of the identity map with the associators on the level of generators. □

1.3 Quadratic Operads

In this subsection, we are going to deal with operadic analogues of the constructions and phenomena we have seen in the previous subsection. For this we will be dealing with quadratic operads which were introduced in [GK94]. Since some of our definitions are slightly different from the usual ones, we will be more explicit than while we were working with quadratic algebras even though everything is in direct analogy with their counterparts in the usual theory of quadratic duality for quadratic operads.

Definition 1.24.

- A *(binary) quadratic operad* is an operad $(\mathscr{P}(n))_{n \in N}$ which is a quotient of a free operad $\mathfrak{F}(E)$ on a Σ-module E in the category of finite dimensional k-vector spaces which is concentrated in degree 2 (i. e. $E(n) \cong 0$ for $n \neq 2$) by an operadic ideal generated by a subrepresentation of relations $R \subseteq \mathfrak{F}(E)(3)$.

- The category **QO** of quadratic operads has as objects quadratic operads and as morphisms morphisms of operads, i. e. morphisms of Σ-modules compatible with operadic compositions.

Notation 1.25. Given a quadratic operad $\mathscr{P}$, $\mathscr{P}(2)$ is its space of generators. We will denote its representation of relations by $\mathfrak{R}(\mathscr{P}) \subseteq \mathscr{P}(3)$.

Given a finite dimensional representation V of Σ_2, we will abuse notation and identify if with the Σ-module E with $E(2) = V$ and $E(n) = 0$ for $n \neq 2$. We will denote the quadratic operad generated by V with relations $R \subseteq \mathfrak{F}(V)(3)$ by $\mathfrak{P}(V, R)$.

Remark 1.26. Given quadratic operads $\mathscr{P}$ and $\mathbb{Q}$, restriction to the spaces of generators induces a bijection

$$\mathrm{Hom}_{\mathbf{QO}}(\mathscr{P}, \mathbb{Q}) \cong \{f \in \mathrm{Hom}_{\Sigma_2}(\mathscr{P}(2), \mathbb{Q}(2)) \mid (\mathfrak{F}(f)(3))(\mathfrak{R}(\mathscr{P})) \subseteq \mathfrak{R}(\mathbb{Q})\}$$

by the universal property of free operads and quotient operads.

Next, we will need some conventions and remarks about representations and their tensor products.

Definition 1.27. Given a representation V of the symmetric group Σ_n, $n \in \mathbb{N}$, its *dual* V^* has as underlying vector space $\mathrm{Hom}_k(V, k)$ and each $\sigma \in \Sigma_n$ acts via $(\sigma \cdot f)(v) = f(\sigma^{-1} \cdot v)$.

Remark 1.28. Given a subrepresentation W of a Σ_n-representation V, the k-subspace $W^\perp \subseteq V^*$ is Σ_n-invariant, thus naturally a Σ_n-representation.

Remark 1.29. Given representations V, W of Σ_n, $V \otimes W$ can be endowed with a natural action of Σ_n via $\sigma \cdot (v \otimes w) = (\sigma \cdot v) \otimes (\sigma \cdot w)$ for $\sigma \in \Sigma_n$.

Remark 1.30. For all Σ_n-representations V and V the natural k-linear evaluation maps

$$\begin{aligned} V &\to (V^*)^* \\ v &\mapsto (f \mapsto f(v)) \end{aligned}$$

and

$$\begin{aligned} V^* \otimes W^* &\to (V \otimes W)^* \\ f \otimes g &\mapsto (v \otimes w \mapsto f(v) \cdot g(w)) \end{aligned}$$

are Σ_n-equivariant.

In particular if the representations in question are finite dimensional, we obtain isomorphisms $V \cong (V^*)^*$ and $V^* \otimes W^* \cong (V \otimes W)^*$ of representations. In the following, whenever we write an isomorphism between such representations, we will be using the above mentioned evaluation map.

Convention 1.31. Let V be a Σ_2-representation. Note that we have an isomorphism

$$\mathfrak{F}(V)(3) \cong (V \otimes V) \oplus (V \otimes V) \oplus (V \otimes V)$$

of k-vector spaces, where the summands correspond to the three composition schemes for producing a (symmetric) ternary operation from two (symmetric) binary operations. In the language of graftings, these schemes are given as follows:

$$\begin{aligned} (\mu \circ_1 \nu)(a,b,c) &= \mu(\nu(a,b),c), \\ (\mu \circ_2 \nu)(a,b,c) &= \mu(a,\nu(b,c)), \\ ((\sigma_{2,3})(\mu \circ_1 \nu))(a,b,c) &= \mu(\nu(a,c),b), \end{aligned}$$

where $\sigma_{2,3} \in \Sigma_3$ is the permutation which swaps 2 and 3.

We will denote the collection of these composition schemes by $\mathscr{C}$ and use the symbols $\circ_t$, $\circ_u$ etc. to refer to generic elements of this collection.

Remark 1.32. Let V be a finite dimensional Σ_2-representation. Then, using the description of $\mathfrak{F}(V)(3)$ and $\mathfrak{F}(V^*)(3)$ as in Convention 1.31, the evaluation morphisms of Remark 1.30 yield an isomorphism

$$\begin{aligned} \mathfrak{F}(V^*)(3) &\xrightarrow{\cong} \mathfrak{F}(V)(3)^* \\ f \circ_t f' &\mapsto \left(v \circ_u v' \mapsto \begin{cases} f(v) \cdot f'(v') & \circ_t = \circ_u \\ 0 & \text{otherwise} \end{cases} \right) \end{aligned}$$

which is Σ_3-equivariant. In the following, whenever we write an isomorphism between a representation of the form $\mathfrak{F}(V^*)(3)$ and a representation of the form $\mathfrak{F}(V^*)(3)$, we will be using this map.

Now we move on to define quadratic duality of operads. Our notion of quadratic duality differs from the one in [GK94] because we do not twist dual representations by the sign representation. With the original definition, quadratic operads still would not form an r-category since an analogue of Proposition 1.48 would apply, but the non-existence of a comparison morphism from the black product to the white product (cf. Proposition 1.49) would merely be a sign issue on the level of generators. We use the non-twisted version to emphasize that there is (also) a problem on the level of relations.

Definition 1.33. The *quadratic dual* construction on **QO** is given by

$$\blacksquare^! = \mathfrak{P}((\blacksquare(2))^*, \mathfrak{R}(\blacksquare)^\perp)$$

where we identify $\mathfrak{R}(\blacksquare)^\perp \subseteq \mathfrak{F}(\blacksquare)(3)^*$ with a subrepresentation of $\mathfrak{F}(\blacksquare^*)(3)$ via the isomorphism of Remark 1.32.

Remark 1.34. As in the case of quadratic algebras, the quadratic dual construction defines an equivalence $\mathbf{QO} \to \mathbf{QO}^{\mathrm{op}}$. Indeed, identifying the double dual of a finite representation with itself and the "double complement" of a subrepresentation with itself, we see that $((\blacksquare)^!)^! \cong \blacksquare$.

As in the case of quadratic algebras, we will see in Corollary 1.47 that this equivalence is in fact induced by a dualizing object with respect to a monoidal structure on $\mathbf{QO}$.

We are now going to define black and white products of quadratic operads. These were introduced in [GK94] and their definition was corrected in [GK95]. A more conceptual treatment which can also be applied to e. g. properads can be found in [Val08]. Similar to the case of quadratic duality, our definition of the black product differs from the usual ones by a sign.

We begin by defining some auxiliary maps.

Definition 1.35. Let V, W be finite dimensional representations of Σ_2.

- The natural morphism

$$\phi^{V,W} : \mathfrak{F}(V \otimes W)(3) \to \mathfrak{F}(V)(3) \otimes \mathfrak{F}(W)(3)$$

  is given by the fact that the tensor product of a $\mathfrak{F}(V)$-algebra A with a $\mathfrak{F}(W)$-algebra B has the structure of a $\mathfrak{F}(V \otimes W)$-algebra which is, on the level of operations, given by the formula

$$(\mu \otimes \nu)(a_1 \otimes b_1, a_2 \otimes b_2, a_3 \otimes b_3) = \mu(a_1, a_2, a_3) \otimes \nu(b_1, b_2, b_3)$$

  for $\mu \in \mathfrak{F}(V)(3)$, $\nu \in \mathfrak{F}(W)(3)$, $a_1, a_2, a_3 \in A$, $b_1, b_2, b_3 \in B$. In the description of Convention 1.31, this means that for $v, v' \in V$, $w, w' \in W$ and $\circ_t \in \mathscr{C}$ we have

$$\phi((v \otimes w) \circ_t (v' \otimes w')) = (v \circ_t v') \otimes (w \circ_t w').$$

- The morphism $\psi^{V,W}$ is given by the composition

$$\begin{aligned} \mathfrak{F}(V)(3) \otimes \mathfrak{F}(W)(3) &\cong \mathfrak{F}(V^*)(3)^* \otimes \mathfrak{F}(W^*)(3)^* \\ &\cong (\mathfrak{F}(V^*)(3) \otimes \mathfrak{F}(W^*)(3))^* \\ &\xrightarrow{(\phi^{V^*,W^*})^*} \mathfrak{F}(V^* \otimes W^*)(3)^* \\ &\cong \mathfrak{F}((V \otimes W)^*)(3)^* \\ &\cong \mathfrak{F}(V \otimes W)(3). \end{aligned}$$

  In particular, in the description of Convention 1.31, we have

$$\psi((v \circ_t v') \otimes (w \circ_u w')) = \begin{cases} (v \otimes w) \circ_t (v' \otimes w') & \circ_t = \circ_u \\ 0 & \text{otherwise} \end{cases}$$

  for $v, v' \in V$, $w, w' \in W$ and $\circ_t, \circ_u \in \mathscr{C}$.

Remark 1.36. Let V, W be finite dimensional representations of Σ_2. Then, $\psi^{V,W}$ is surjective since elements of the form $(v \otimes w) \circ_t (v' \otimes w')$ for $v, v' \in V$, $w, w' \in W$, $\circ_t \in \mathscr{C}$ generate $\mathfrak{F}(V \otimes W)(3)$ and are in the image of $\psi^{V,W}$.

Thus, in particular, $\phi^{V,W}$ is injective.

We can now define black and white products for quadratic operads.

Definition 1.37. The *black product* of quadratic operads is defined as

$$\blacksquare \bullet \blacklozenge = \mathfrak{P}((\blacksquare(2)) \otimes (\blacklozenge(2)), \psi(\mathfrak{R}(\blacksquare) \otimes \mathfrak{R}(\blacklozenge))).$$

With this definition of the black product, we also have to slightly modify its unit object.

Definition 1.38. Let k_{triv} be the trivial 1-dimensional representation of Σ_2. Let $R_{\mathscr{L}} \subseteq \mathfrak{F}(k_{\mathrm{triv}})(3)$ be the 1-dimensional subrepresentation generated by the "Jacobi relation"

$$J := l(x_1, l(x_2, x_3)) + l(x_2, l(x_3, x_1)) + l(x_3, l(x_1, x_2))$$

for a generating binary operation l which corresponds to $1 \in k_{\mathrm{triv}}$.

With these notations, we define $\mathscr{L} := \mathfrak{P}(k_{\mathrm{triv}}, R_{\mathscr{L}})$.

Remark 1.39. Let V be Σ_2-representation, $R \subseteq \mathfrak{F}(V)(3)$ a subrepresentation. Consider the natural isomorphism $\rho\colon V \otimes k_{\mathrm{triv}} \cong V$ given by the right unitor of the monoidal structure on σ_2-representations. Then, since $\psi((v \circ_t w) \otimes J) = (v \otimes l) \circ_t (w \otimes l)$ for all $v, w \in V$ and $\circ_t \in \mathscr{C}$, the isomorphism $\mathfrak{F}(\rho)(3)\colon \mathfrak{F}(V \otimes k_{\mathrm{triv}})(3) \cong \mathfrak{F}(V)(3)$ maps $\psi(R \otimes R_{\mathscr{L}})$ isomorphically to R. Thus, by Remark 1.26, we have a natural isomorphism $\mathfrak{P}(V, R) \bullet \mathscr{L} \cong \mathfrak{P}(V, R)$ of quadratic operads.

In fact, also the left unitor, the associator and the braiding of $(\Sigma_2 - \mathrm{Rep}_k, \otimes, k_{\mathrm{triv}})$ extend in a similar way to quadratic operads to yield a symmetric monoidal structure on **QO** with $\bullet$ as the monoidal product and $\mathscr{L}$ as the monoidal unit.

Definition 1.40. The *white product* of quadratic operads is defined as

$$\blacksquare \circ \blacklozenge = \mathfrak{P}((\blacksquare(2)) \otimes (\blacklozenge(2)), \phi^{-1}(\mathfrak{R}(\blacksquare) \otimes \mathfrak{F}(\blacklozenge(2))(3) + \mathfrak{F}(\blacksquare(2))(3) \otimes \mathfrak{R}(\blacklozenge))).$$

Definition 1.41. Let k_{triv} be the trivial 1-dimensional representation of Σ_2. Let $R_{\mathscr{Com}} \subseteq \mathfrak{F}(k_{\mathrm{triv}})(3)$ be the 2-dimensional subrepresentation generated by the "associativity relations"

$$A_1 := m(x_1, m(x_2, x_3)) - m(x_2, m(x_3, x_1)),$$
$$A_2 := m(x_2, m(x_3, x_1)) - m(x_3, m(x_1, x_2))$$

for a generating binary operation m which corresponds to $1 \in k_{\mathrm{triv}}$.

With these notations, we define $\mathscr{Com} := \mathfrak{P}(k_{\mathrm{triv}}, R_{\mathscr{L}})$. The name $\mathscr{Com}$ comes from the fact that $\mathscr{Com}$ classifies commutative algebras.

Remark 1.42. Let V be Σ_2-representation, $R \subseteq \mathfrak{F}(V)(3)$ a subrepresentation. Consider the natural isomorphism $\rho\colon V \otimes k_{\text{triv}} \cong V$ given by the right unitor of the monoidal structure on σ_2-representations. Then, since the preimage of $R \otimes \mathfrak{F}(k_{\text{triv}})(3) + \mathfrak{F}(V)(3) \otimes R_{\mathscr{C}om}$ under ϕ consists of elements of the form $\sum_{v,v',\circ_t}(v \otimes m) \circ_t (v' \otimes m)$ for $\sum_{v,v',\circ_t} v \circ_t v' \in R$, the isomorphism $\mathfrak{F}(\rho)(3)\colon \mathfrak{F}(V \otimes k_{\text{triv}})(3) \cong \mathfrak{F}(V)(3)$ maps $\phi^{-1}(R \otimes \mathfrak{F}(k_{\text{triv}})(3) + \mathfrak{F}(V)(3) \otimes R_{\mathscr{C}om})$ isomorphically to R. Thus, by Remark 1.26, we have a natural isomorphism $\mathfrak{P}(V, R) \circ \mathscr{C}om \cong \mathfrak{P}(V, R)$ of quadratic operads.

In fact, also the left unitor, the associator and the braiding of $(\Sigma_2 - \text{Rep}_k, \otimes, k_{\text{triv}})$ extend in a similar way to quadratic operads to yield a symmetric monoidal structure on **QO** with $\circ$ as the monoidal product and $\mathscr{C}om$ as the monoidal unit.

Now we want to show that, as for quadratic algebras, this monoidal structure is essentially the one obtained by transferring the black product along the quadratic dual construction (as in Remark 1.2).

Lemma 1.43. *Let V and W be finite dimensional Σ_2-representations. Further let $\alpha \in \mathfrak{F}(V^*)(3) \otimes \mathfrak{F}(W^*)(3)$ and $\beta \in \mathfrak{F}(V \otimes W)(3)$. Consider α as an element of $(\mathfrak{F}(V)(3) \otimes \mathfrak{F}(W)(3))^*$ and $\psi(\alpha) \in \mathfrak{F}(V^* \otimes W^*)(3)$ as an element of $\mathfrak{F}(V \otimes W)(3)^*$ via the identifications of Remark 1.30 and Remark 1.32. Then we have $\alpha(\phi(\beta)) = (\psi(\alpha))(\beta)$.*

Proof. Using Convention 1.31, it is enough to show the statement for $\alpha = (f \circ_t f') \otimes (g \circ_u g')$ and $\beta = (v \otimes w) \circ_s (v' \otimes w')$ where $v, v' \in V$, $w, w' \in W$, $f, f' \in V^*$, $g, g' \in W^*$ and $\circ_s, \circ_t, \circ_u \in \mathscr{C}$.

In this case we have

$$\begin{aligned}
\alpha(\phi(\beta)) &= ((f \circ_t f') \otimes (g \circ_u g'))(\phi((v \otimes w) \circ_s (v' \otimes w'))) \\
&= ((f \circ_t f') \otimes (g \circ_u g'))((v \circ_s v') \otimes (w \circ_s w')) \\
&= (f \circ_t f')(v \circ_s v') \cdot (g \circ_u g')(w \circ_s w') \\
&= \begin{cases} f(v)f'(v') \cdot g(w)g'(w') & \circ_t = \circ_s = \circ_u \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
(\psi(\alpha))(\beta) &= (\psi((f \circ_t f') \otimes (g \circ_u g')))((v \otimes w) \circ_s (v' \otimes w')) \\
&= \begin{cases} ((f \otimes g) \circ_t (f' \otimes g'))((v \otimes w) \circ_s (v' \otimes w')) & \circ_t = \circ_u \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} f(v)g(w)f'(v')g'(w') & \circ_t = \circ_u = \circ_s \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

which proves the claim. $\square$

Proposition 1.44. *Let Σ_2-representations V, W and subspaces $R \subseteq \mathfrak{F}(V)(3)$, $S \subseteq \mathfrak{F}(W)(3)$ be given. Then the natural evaluation isomorphism* $\mathrm{ev}\colon V^* \otimes W^* \cong (V \otimes W)^*$ *extends to an isomorphism*

$$\mathfrak{P}(V,R)^! \bullet \mathfrak{P}(W,S)^! \cong (\mathfrak{P}(V,R) \circ \mathfrak{P}(W,S))^!$$

of quadratic operads.

Proof. By Remark 1.26, it is enough to show that under the identification of Remark 1.32, the subrepresentation

$$\mathfrak{F}(\mathrm{ev})(3)(\mathfrak{R}(\mathfrak{P}(V,R)^! \bullet \mathfrak{P}(W,S)^!)) = \mathfrak{F}(\mathrm{ev})(3)(\psi(R^\perp \otimes S^\perp)) \subseteq \mathfrak{F}((V \otimes W)^*)(3)$$

coincides with the subrepresentation

$$\mathfrak{R}((\mathfrak{P}(V,R) \circ \mathfrak{P}(W,S))^!) = \phi^{-1}(R \otimes \mathfrak{F}(W)(3) + \mathfrak{F}(V)(3) \otimes S)^\perp \subseteq \mathfrak{F}(V \otimes W)(3)^*.$$

Now let $\alpha \in \mathfrak{F}(\mathrm{ev})(3)(\psi(R^\perp \otimes S^\perp))$. Since ψ is surjective (cf. Remark 1.36) and $\mathfrak{F}(\mathrm{ev})(3)$ is an isomorphism, we can find an α' with $\alpha = \mathfrak{F}(\mathrm{ev})(3)(\psi(\alpha'))$. Then, using Lemma 1.43 and injectivity of ϕ (Remark 1.36), we obtain

$$\begin{aligned}
\alpha = \mathfrak{F}(\mathrm{ev})(3)(\psi(\alpha')) \in \phi^{-1}(R \otimes \mathfrak{F}(W)(3) + \mathfrak{F}(V)(3) \otimes S)^\perp &\Leftrightarrow \\
\forall \beta \in \phi^{-1}(R \otimes \mathfrak{F}(W)(3) + \mathfrak{F}(V)(3) \otimes S) : \psi(\alpha')(\beta) = 0 &\Leftrightarrow \\
\forall \beta \in \phi^{-1}(R \otimes \mathfrak{F}(W)(3) + \mathfrak{F}(V)(3) \otimes S) : \alpha'(\phi(\beta)) = 0 &\Leftrightarrow \\
\forall \beta' \in R \otimes \mathfrak{F}(W)(3) + \mathfrak{F}(V)(3) \otimes S : \alpha'(\beta') = 0 &\Leftrightarrow \\
\alpha' \in R^\perp \otimes S^\perp&,
\end{aligned}$$

which yields the desired equality. □

Corollary 1.45. *There are natural isomorphisms*

$$\blacksquare^! \bullet \pentagon^! \cong (\blacksquare \circ \pentagon)^!$$

respectively

$$(\blacksquare^! \bullet \pentagon^!)^! \cong \blacksquare \circ \pentagon$$

of functors $\mathbf{QO} \times \mathbf{QO} \to \mathbf{QO}^{\mathrm{op}}$ *respectively* $\mathbf{QO} \times \mathbf{QO} \to \mathbf{QO}$, *i. e.* $\circ$ *can be obtained by transferring* $\bullet$ *along* $!$ *as in Remark 1.2.*

Similar to the case of quadratic algebras, we have an adjunction relating black product, white product and quadratic duality.

Proposition 1.46. *For every quadratic operad* $\mathscr{P}$, *the functor* $\blacksquare \bullet \mathscr{P}$ *is left adjoint to* $\mathscr{P}^! \circ \blacksquare$.

Proof. Let quadratic operads $\mathscr{P} = \mathfrak{P}(V,R)$, $\mathbb{Q} = \mathfrak{P}(W,S)$ and $\mathscr{R} = \mathfrak{P}(X,T)$ be given. Using Remark 1.26, we want to show that the natural isomorphism

$$\mathrm{Hom}_{\Sigma_2}(W \otimes V, X) \cong \mathrm{Hom}_{\Sigma_2}(W, V^* \otimes X) \tag{1.2}$$

on the level of generators can be extended to an isomorphism

$$\mathrm{Hom}_{\mathbf{QO}}(\mathbb{Q} \bullet \mathscr{P}, \mathscr{R}) \cong \mathrm{Hom}_{\mathbf{QO}}\left(\mathbb{Q}, \mathscr{P}^! \circ \mathscr{R}\right).$$

Now let $f \in \mathrm{Hom}_{\Sigma_2}(W \otimes V, X)$ be given. Let $\hat{f} \in \mathrm{Hom}_{\Sigma_2}(W, V^* \otimes X)$ be its adjoint. First we want to analyze some evaluations. Let $v, v' \in V$, $w, w' \in W$, $g, g' \in X^*$ and $\circ_s, \circ_t, \circ_u \in \mathscr{C}$. Then, under the identifications of Remark 1.30 and Remark 1.32, we have

$$\begin{aligned}
&(\psi((w \circ_s w') \otimes (g \circ_t g')))(\mathfrak{F}(\hat{f})(3)(v \circ_u v')) \\
&= \begin{cases} ((w \otimes g) \circ_t (w' \otimes g'))(\mathfrak{F}(\hat{f})(3)(v \circ_u v')) & \circ_s = \circ_t \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} (\hat{f}(v))(w \otimes g) \cdot (\hat{f}(v'))(w' \otimes g') & \circ_s = \circ_t = \circ_u \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} g(f(v \otimes w)) \cdot g'(f(v' \otimes w')) & \circ_s = \circ_t = \circ_u \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
&(g \circ_t g')(\mathfrak{F}(f)(3)(\psi((v \circ_u v') \otimes (w \circ_s w')))) \\
&= \begin{cases} (g \circ_t g')(\mathfrak{F}(f)(3)((v \otimes w) \circ_u (v' \otimes w'))) & \circ_s = \circ_u \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} g(f(v \otimes w)) \cdot g'(f(v' \otimes w)) & \circ_s = \circ_u = \circ_t \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Thus we see that

$$(\psi((w \circ_s w') \otimes (g \circ_t g')))(\mathfrak{F}(\hat{f})(3)(v \circ_u v')) = (g \circ_t g')(\mathfrak{F}(f)(3)(\psi((v \circ_u v') \otimes (w \circ_s w')))). \tag{1.3}$$

Now note that the inclusion

$$\mathfrak{F}(\hat{f})(3)(R) \subseteq \mathfrak{R}(\mathscr{P}^! \circ \mathscr{R}) = \phi^{-1}(S^\perp \otimes \mathfrak{F}(X)(3) + \mathfrak{F}(W^*)(3) \otimes T) \tag{1.4}$$

holds if and only if for all $\alpha \in \phi^{-1}(S^\perp \otimes \mathfrak{F}(X)(3) + \mathfrak{F}(W^*)(3) \otimes T)^\perp$ and $\beta \in R$ we have $\alpha(\mathfrak{F}(\hat{f})(3)(\beta)) = 0$. By Proposition 1.44 we can identify $\phi^{-1}(S^\perp \otimes \mathfrak{F}(X)(3) + \mathfrak{F}(W^*)(3) \otimes T)^\perp$ with $\psi(S \otimes T^\perp)$, so the inclusion (1.4) holds if and only if for all $\beta \in R$, $\gamma \in S$ and $\delta \in T^\perp$ we have $(\psi(\gamma \otimes \delta))(\mathfrak{F}(\hat{f})(3)(\beta)) = 0$.

Now by (1.3), under suitable identifications of duals, for $\beta \in R$, $\gamma \in S$ and $\delta \in T^{\perp}$ we have

$$(\psi(\gamma \otimes \delta))(\mathfrak{F}(\hat{f})(3)(\beta)) = \delta(\mathfrak{F}(f)(3)(\psi(\beta \otimes \gamma))).$$

Thus the inclusion (1.4) holds if and only if for all $\beta \in R$, $\gamma \in S$ and $\delta \in T^{\perp}$ we have $\delta(\mathfrak{F}(f)(3)(\psi(\beta \otimes \gamma))) = 0$, i. e. $\mathfrak{F}(f)(3)(\psi(S \otimes R)) \subseteq T$. Hence the the adjunction isomorphism of (1.2) can indeed be extended to quadratic operads. □

With this adjunction at hand, we can put the quadratic duality of quadratic operads into the context of GV categories as it was done in [Man17].

Corollary 1.47. *The unit $\mathscr{C}om$ of the white product of quadratic operad is a dualizing object for the black product of quadratic operads since for all $\mathscr{P} \in$ **QO**, the functor $\mathrm{Hom}_{\mathbf{QO}}(\blacksquare \bullet \mathscr{P}, \mathscr{C}om) \cong \mathrm{Hom}_{\mathbf{QO}}\left(\blacksquare, \mathscr{P}^! \circ \mathscr{C}om\right)$ is representable by $\mathscr{P}^! \circ \mathscr{C}om \cong \mathscr{P}^!$.*

Now we will show that, as in the case of quadratic algebras, **QO** together with the black product is not an r-category.

Proposition 1.48. *The monoidal unit $\mathscr{L}$ of the black product of quadratic operads is not a dualizing object, i. e. there is no equivalence $D\colon \mathbf{QO} \to \mathbf{QO}^{\mathrm{op}}$ s. t. for all $\mathscr{P} \in \mathbf{QO}$ the functor $\mathrm{Hom}_{\mathbf{QO}}(\blacksquare \bullet \mathscr{P}, \mathscr{L})$ is represented by $D(\mathscr{P})$.*

Proof. The adjunction of Proposition 1.46 yields for all $\mathscr{P}, \mathbb{Q} \in \mathbf{QO}$ natural isomorphisms

$$\mathrm{Hom}_{\mathbf{QO}}(\mathbb{Q} \bullet \mathscr{P}, \mathscr{L}) \cong \mathrm{Hom}_{\mathbf{QO}}\left(\mathbb{Q}, \mathscr{P}^! \circ \mathscr{L}\right).$$

Thus, if $\mathscr{L}$ were a dualizing object, $D(\mathscr{P})$ would be isomorphic to $\mathscr{P}^! \circ \mathscr{L}$ by the uniqueness of representing objects.

Note that we have

$$\begin{aligned}
&\mathfrak{P}(k_{\mathrm{triv}}, \mathfrak{F}(k_{\mathrm{triv}})(3))^! \circ \mathscr{L} \\
\cong &\mathfrak{P}(k^*_{\mathrm{triv}} \otimes k_{\mathrm{triv}}, \phi^{-1}(0 \otimes \mathfrak{F}(k_{\mathrm{triv}})(3) + \mathfrak{F}(k^*_{\mathrm{triv}})(3) \otimes \mathfrak{R}(\mathscr{L}))) \\
= &\mathfrak{P}(k^*_{\mathrm{triv}} \otimes k_{\mathrm{triv}}, \phi^{-1}(\mathfrak{F}(k^*_{\mathrm{triv}})(3) \otimes \mathfrak{R}(\mathscr{L}))).
\end{aligned}$$

Now the elements which are in the image of ϕ are of the form $\sum_{\circ_t \in \mathscr{C}} c_{\circ_t} \cdot (f \circ_t l) \otimes (f \circ_t l)$ where f is a generator of k^*_{triv} and $c_{\circ_t} \in k$, whereas non-trivial elements of $\mathfrak{F}(k^*_{\mathrm{triv}})(3) \otimes \mathfrak{R}(\mathscr{L})$ always have summands with different composition schemes in their two tensor factors. Thus we obtain $\phi^{-1}(\mathfrak{F}(k^*_{\mathrm{triv}})(3) \otimes \mathfrak{R}(\mathscr{L})) = 0$ and hence

$$\mathfrak{P}(k_{\mathrm{triv}}, \mathfrak{F}(k_{\mathrm{triv}})(3))^! \circ \mathscr{L} \cong \mathfrak{P}(k^*_{\mathrm{triv}} \otimes k_{\mathrm{triv}}, 0) \cong \mathfrak{F}(k_{\mathrm{triv}}).$$

Now using Remark 1.34, Corollary 1.45 and Remark 1.42 we obtain

$$\mathscr{Com}^! \bullet \blacksquare \cong \mathscr{Com}^! \bullet (\blacksquare^!)^! \cong (\mathscr{Com} \circ \blacksquare^!)^! \cong (\blacksquare^!)^! \cong \blacksquare$$

and similarly $\blacksquare \bullet \mathscr{Com}^! \cong \blacksquare$. Thus, by the uniqueness of monoidal units, we have $\mathscr{Com}^! \cong \mathscr{L}$. Hence we obtain

$$\begin{aligned}
&\mathscr{Com}^! \circ \mathscr{L} \\
\cong &\mathscr{L} \circ \mathscr{L} \\
\cong &\mathfrak{P}(k_{\mathrm{triv}} \otimes k_{\mathrm{triv}}, \phi^{-1}(\mathfrak{R}(\mathscr{L}) \otimes \mathfrak{F}(k_{\mathrm{triv}})(3) + \mathfrak{F}(k_{\mathrm{triv}})(3) \otimes \mathfrak{R}(\mathscr{L}))).
\end{aligned}$$

As above, elements in the image of ϕ are of the form $\sum_{\circ_t \in \mathscr{C}} c_{\circ_t} \cdot (l \circ_t l) \otimes (l \circ_t l)$, but elements of $\mathfrak{R}(\mathscr{L}) \otimes \mathfrak{F}(k_{\mathrm{triv}})(3) + \mathfrak{F}(k_{\mathrm{triv}})(3) \otimes \mathfrak{R}(\mathscr{L})$ always have summands with different composition schemes in their two tensor factors. Thus we obtain $\phi^{-1}(\mathfrak{R}(\mathscr{L}) \otimes \mathfrak{F}(k_{\mathrm{triv}})(3) + \mathfrak{F}(k_{\mathrm{triv}})(3) \otimes \mathfrak{R}(\mathscr{L})) = 0$ and hence

$$\mathscr{Com}^! \circ \mathscr{L} \cong \mathfrak{P}(k_{\mathrm{triv}} \otimes k_{\mathrm{triv}}, 0) \cong \mathfrak{F}(k_{\mathrm{triv}}).$$

This means that the assignment $\mathscr{P} \mapsto \mathscr{P}^! \circ \mathscr{L}$ maps the non-isomorphic objects $\mathfrak{P}(k_{\mathrm{triv}}, \mathfrak{F}(k_{\mathrm{triv}})(3))$ and $\mathscr{Com}$ onto isomorphic objects. Thus there cannot be an equivalence of categories that is given by this assignment on the objects. $\square$

In this situation, one may hope to mimic the construction of Proposition 1.23 to obtain a natural transformation $\bullet \Rightarrow \circ$ which is compatible with the associators, but this approach does not work for quadratic operads.

Proposition 1.49. *The natural transformation* $\mathrm{id}_\otimes$ *from the tensor product functor for* Σ_2*-representations to itself cannot be extended to a natural transformation* $\bullet \Rightarrow \circ$.

Proof. If that were the case, then we would in particular have a morphism of quadratic operads

$$F\colon \mathscr{Com} \cong \mathscr{Com} \bullet \mathscr{L} \to \mathscr{Com} \circ \mathscr{L} \cong \mathscr{L}$$

which restricts to $\mathrm{id}_{k_{\mathrm{triv}}}\colon k_{\mathrm{triv}} \to k_{\mathrm{triv}}$ on generators. This map on the generators extends to a morphism of operads $\mathscr{Com} \to \mathscr{L}$ if and only if $\mathfrak{R}(\mathscr{Com}) \subseteq \mathfrak{R}(\mathscr{L})$, which cannot be the case since $\dim(\mathfrak{R}(\mathscr{Com})) = 2$ whereas $\dim(\mathfrak{R}(\mathscr{L})) = 1$. $\square$

2 The Gravity Operad

The gravity operad, which was first introduced in [Get94] and in [GK94], has several equivalent definitions. In this section we are going to construct an incarnation of it using the homology of the moduli spaces $\mathfrak{M}_n$ of smooth n-marked genus 0 curves (cf. Definition B.1). This approach, which goes back to [GK94] and [Get95], also paves the path to establishing the Koszul duality between the gravity operad and the hypercommutative operad (cf. Theorem 2.7).

2.1 The Gravity Cooperad

Since the residue maps which induce the (co)operadic (co)compositions of the gravity (co)operad are better described in terms of cohomology, we will first construct the gravity *cooperad* using the cohomology of $\mathfrak{M}_n$ and dualize this definition to obtain the gravity operad.

Definition 2.1. Let $\mathscr{CoGrav}$ be the Σ-module in the category of graded $\mathbb{C}$-vector spaces with

$$\mathscr{CoGrav}(n) := \det(\mathbb{C}^n) \otimes H^{\bullet-1}(\mathfrak{M}_{n+1})$$

where the Σ_n-action on $\det(\mathbb{C}^n)$ is induced by permuting the coordinates of $\mathbb{C}^n$ and the Σ_n-action on $H^{\bullet-1}(\mathfrak{M}_{n+1})$ is induced by the action on $\mathfrak{M}_{n+1}$ given by permuting the first n marked points.

Definition 2.2. Let $k, l \in \mathbb{N}_+$, $i \in \{1, \ldots, k\}$. We define the infinitesimal cocomposition morphism $\Delta_i \colon \mathscr{CoGrav}(k+l-1) \to \mathscr{CoGrav}(k) \otimes \mathscr{CoGrav}(l)$ of the gravity cooperad as follows:

Let e denote the unique internal edge of $C_{k+1} \circ_i C_{l+1}$. Let the bijection $f_i \colon \{1, \ldots, k+l-1\} \dot\cup \{e\} \cong \{1, \ldots, k\} \dot\cup \{1, \ldots, l\}$ be given by

$$f_i(x) = \begin{cases} i \in \{1, \ldots, k\} & x = e \\ j \in \{1, \ldots, k\} & x = j \in \{1, \ldots, k+l-1\}, j < i \\ j - i + 1 \in \{1, \ldots, l\} & x = j \in \{1, \ldots, k+l-1\}, i \leqslant j < i + l \\ j - l + 1 \in \{1, \ldots, k\} & x = j \in \{1, \ldots, k+l-1\}, i + l \leqslant j \end{cases}.$$

Let

$$\phi_i \colon \det(\mathbb{C}^{k+l-1}) \otimes \det(\mathbb{C}^{\{e\}}) \xrightarrow{\cong} \det(\mathbb{C}^k) \otimes \det(\mathbb{C}^l)$$

be the isomorphism induced by f_i.

Now the infinitesimal cocomposition morphism Δ_i is defined as the composition

$$
\begin{aligned}
&& \mathscr{CoGrav}(k+l-1) = \det(\mathbb{C}^{k+l-1}) \otimes H^{\bullet-1}(\mathfrak{M}_{k+l}) \\
&\xrightarrow{\mathrm{id}\otimes \mathrm{Res}_{C_{k+1}\circ_i C_{l+1}, C_{k+l}}} & \det(\mathbb{C}^{k+l-1}) \otimes \det(\mathbb{C}^{\{e\}}) \otimes H^{\bullet-2}(\mathfrak{M}(C_{k+1} \circ_i C_{l+1})) \\
&\xrightarrow{\mathrm{id}\otimes \Psi^*_{C_{k+1},C_{l+1},i}} & \det(\mathbb{C}^{k+l-1}) \otimes \det(\mathbb{C}^{\{e\}}) \otimes H^{\bullet-2}(\mathfrak{M}_{k+1} \times \mathfrak{M}_{l+1}) \\
&\xrightarrow{\phi_i \otimes \mathrm{id}} & \det(\mathbb{C}^{k}) \otimes \det(\mathbb{C}^{l}) \otimes H^{\bullet-2}(\mathfrak{M}_{k+1} \times \mathfrak{M}_{l+1}) \\
&\xrightarrow{\mathrm{id}\otimes\kappa} & \det(\mathbb{C}^{k}) \otimes \det(\mathbb{C}^{l}) \otimes H^{\bullet-1}(\mathfrak{M}_{k+1}) \otimes H^{\bullet-1}(\mathfrak{M}_{l+1}) \\
&\xrightarrow{\mathrm{id}\otimes\tau\otimes\mathrm{id}} & \det(\mathbb{C}^{k}) \otimes H^{\bullet-1}(\mathfrak{M}_{k+1}) \otimes \det(\mathbb{C}^{l}) \otimes H^{\bullet-1}(\mathfrak{M}_{l+1}) \\
&= & \mathscr{CoGrav}(k) \otimes \mathscr{CoGrav}(l),
\end{aligned}
\tag{2.1}
$$

where $\Psi_{C_{k+1},C_{l+1},i} \colon \mathfrak{M}_{k+1} \times \mathfrak{M}_{l+1} \to \mathfrak{M}(C_{k+1} \circ_i C_{l+1})$ is the isomorphism of Fact B.9 and Fact B.10.

Proposition 2.3. *The cocomposition maps of Definition 2.2 endow $\mathscr{CoGrav}$ with the structure of a cooperad.*

Proof. Let k, l, i as in Definition 2.2.

Equivariance. We need to show that for $\sigma \in \Sigma_k$, the diagram

$$
\begin{tikzcd}
\mathscr{CoGrav}(k+l-1) \arrow[r, "\Delta_i"] \arrow[d, "\theta_{k,i}(\sigma)"'] & \mathscr{CoGrav}(k) \otimes \mathscr{CoGrav}(l) \arrow[d, "\sigma\otimes\mathrm{id}"] \\
\mathscr{CoGrav}(k+l-1) \arrow[r, "\Delta_{\sigma(i)}"'] & \mathscr{CoGrav}(k) \otimes \mathscr{CoGrav}(l)
\end{tikzcd}
\tag{2.2}
$$

commutes, where $\theta_{k,i} \colon \Sigma_k \hookrightarrow \Sigma_{k+l-1}$ is the inclusion defined in Notation A.12. In order to show this, we will analyze the definition of Δ_i resp. $\Delta_{\sigma(i)}$ step by step and show the compatibility of each map in (2.1) with the action of σ.

First, we note that $C_{k+1} \circ_{\sigma(i)} C_{l+1} \cong \theta_{k,i}(\sigma)(C_{k+1} \circ_i C_{l+1})$. Hence, by Proposition B.21 and with the notation thereof, we have a commutative diagram

$$
\begin{tikzcd}
H^{\bullet-1}(\mathfrak{M}_{k+l}) \arrow[r, "\mathrm{Res}_{C_{k+1}\circ_i C_{l+1}, C_{k+l}}"] \arrow[d, "(\theta_{k,i}(\sigma)^*)^{-1}"'] & \det(\mathbb{C}^{\{e\}}) \otimes H^{\bullet-2}(\mathfrak{M}(C_{k+1} \circ_i C_{l+1})) \arrow[d, "(\mathrm{id}\otimes\theta_{k,i}(\sigma)^*)^{-1}"] \\
H^{\bullet-1}(\mathfrak{M}_{k+l}) \arrow[r, "\mathrm{Res}_{C_{k+1}\circ_{\sigma(i)} C_{l+1}, C_{k+l}}"'] & \det(\mathbb{C}^{\{e\}}) \otimes H^{\bullet-2}(\mathfrak{M}(C_{k+1} \circ_{\sigma(i)} C_{l+1}))
\end{tikzcd}
.
$$

Tensoring the vertical maps with

$$\theta_{k,i}(\sigma) \colon \det(\mathbb{C}^{k+l-1}) \to \det(\mathbb{C}^{k+l-1}),$$

we see that $\mathrm{id}\otimes\mathrm{Res}_{C_{k+1}\circ_i C_{l+1},C_{k+l}}$ and $\mathrm{id}\otimes\mathrm{Res}_{C_{k+1}\circ_{\sigma(i)}C_{l+1},C_{k+l}}$ are compatible with the action of σ.

Next, the compatibility of $\Psi_{C_{k+1},C_{l+1},i}$ resp. $\Psi_{C_{k+1},C_{l+1},\sigma(i)}$ with the strata of $\overline{\mathfrak{M}}_{k+1}\times\overline{\mathfrak{M}}_{l+1}$ (cf. Fact B.9) and the Σ_k-action (cf. Fact B.10) yields a commutative diagram

$$\begin{tikzcd}
\mathfrak{M}(C_{k+1}\circ_i C_{l+1}) \arrow[d, "\theta_{k,i}(\sigma)"'] & \mathfrak{M}_{k+1}\times\mathfrak{M}_{l+1} \arrow[l, "\Psi_{C_{k+1},C_{l+1},i}"'] \arrow[d, "\sigma\times\mathrm{id}"] \\
\mathfrak{M}(C_{k+1}\circ_{\sigma(i)} C_{l+1}) & \mathfrak{M}_{k+1}\times\mathfrak{M}_{l+1} \arrow[l, "\Psi_{C_{k+1},C_{l+1},\sigma(i)}"]
\end{tikzcd}.$$

Now taking cohomology and then tensoring the vertical maps with

$$\theta_{k,i}(\sigma)\otimes\phi_{\theta_{k,i}(\sigma)}\colon\ \det(\mathbb{C}^{k+l-1})\otimes\det(\mathbb{C}^{\{e\}})\to\det(\mathbb{C}^{k+l-1})\otimes\det(\mathbb{C}^{\{\theta_{k,i}(\sigma)(e)\}})$$

yields the compatibility of $\mathrm{id}\otimes\Psi^*_{C_{k+1},C_{l+1},i}$ and $\mathrm{id}\otimes\Psi^*_{C_{k+1},C_{l+1},\sigma(i)}$ with the action of σ.

Moreover, we have a commutative diagram

$$\begin{tikzcd}
\{1,\dots,k+l-1\}\dot\cup\{e\} \arrow[r, "f_i"] \arrow[d, "\theta_{k,i}(\sigma)\dot\cup\mathrm{id}"'] & \{1,\dots,k\}\dot\cup\{1,\dots,l\} \arrow[d, "\sigma\dot\cup\mathrm{id}"] \\
\{1,\dots,k+l-1\}\dot\cup\{e\} \arrow[r, "f_{\sigma(i)}"'] & \{1,\dots,k\}\dot\cup\{1,\dots,l\}
\end{tikzcd}$$

of sets coming from identifications of edges. Applying $\det(\mathbb{C}^{\blacksquare})$ to this square and tensoring the vertical maps with

$$((\sigma\times\mathrm{id})^*)^{-1}\colon H^{\bullet-2}(\mathfrak{M}_{k+1}\times\mathfrak{M}_{l+1})\to H^{\bullet-2}(\mathfrak{M}_{k+1}\times\mathfrak{M}_{l+1})$$

yields the compatibility of $\phi_i\otimes\mathrm{id}$ and $\phi_{\sigma(i)}\otimes\mathrm{id}$ with σ.

Finally, the compatibility of $\mathrm{id}\otimes\kappa$ resp. $\mathrm{id}\otimes\tau\otimes\mathrm{id}$ with the corresponding actions of σ on their domains and targets follow directly from the naturality properties of the Künneth isomorphism resp. the braiding. Thus, combining the previous compatibility relations, we see that (2.2) indeed commutes.

The other equivariance relation is given by the commutativity of

$$\begin{tikzcd}
\mathscr{CoGrav}(k+l-1) \arrow[r, "\Delta_i"] \arrow[d, "\vartheta_{l,i}(\rho)"'] & \mathscr{CoGrav}(k)\otimes\mathscr{CoGrav}(l) \arrow[d, "\mathrm{id}\otimes\rho"] \\
\mathscr{CoGrav}(k+l-1) \arrow[r, "\Delta_i"'] & \mathscr{CoGrav}(k)\otimes\mathscr{CoGrav}(l)
\end{tikzcd}$$

for all $\rho\in\Sigma_l$, where $\vartheta_{l,i}\colon\Sigma_l\hookrightarrow\Sigma_{k+l-1}$ is the inclusion defined in Notation A.12. Using the fact that $C_{k+1}\circ_i C_{l+1}\cong\vartheta_{l,i}(\rho)(C_{k+1}\circ_i C_{l+1})$, this can be shown with a similar analysis as for the compatibility relation for $\sigma\in\Sigma_k$.

Cocomposition axioms. Let $m\in\mathbb{N}_+$ and $j\in\{i+1,\dots,k\}$. The parallel cocomposition axiom requires that the diagram

$$\begin{tikzcd}
\mathscr{CoGrav}(k+l+m-2) \arrow[r, "\Delta_i"] \arrow[dd, "\Delta_{j+m-1}"'] & \mathscr{CoGrav}(k+m-1)\otimes\mathscr{CoGrav}(l) \arrow[d, "\Delta_j\otimes\mathrm{id}"] \\
& \mathscr{CoGrav}(k)\otimes\mathscr{CoGrav}(m)\otimes\mathscr{CoGrav}(l) \arrow[d, "\mathrm{id}\otimes\tau"] \\
\mathscr{CoGrav}(k+l-1)\otimes\mathscr{CoGrav}(m) \arrow[r, "\Delta_i\otimes\mathrm{id}"'] & \mathscr{CoGrav}(k)\otimes\mathscr{CoGrav}(l)\otimes\mathscr{CoGrav}(m)
\end{tikzcd} \tag{2.3}$$

commutes.

First we note that $(C_{k+1}\circ_j C_{m+1})\circ_i C_{l+1}$ and $(C_{k+1}\circ_i C_{l+1})\circ_{j+l-1} C_{m+1}$ represent the same isomorphism class $T\in\mathcal{T}_{k+l+m-1}$. Let $e\in\mathrm{In}(T)$ be the edge which is used in grafting C_{k+1} with C_{l+1} in the grafting scheme $(C_{k+1}\circ_i C_{l+1})\circ_{j+l-1} C_{m+1}$ and $f\in\mathrm{In}(T)$ the edge which is used in grafting C_{k+1} with C_{m+1} in the grafting scheme $(C_{k+1}\circ_j C_{m+1})\circ_i C_{l+1}$.

Next, Proposition B.23 implies that the compositions

$$\begin{aligned}
& H^{\bullet-1}(\mathfrak{M}_{k+l+m-1}) \\
\xrightarrow{\mathrm{id}\otimes\mathrm{Res}_{C_{k+m}\circ_i C_{l+1},C_{k+l+m-1}}} \quad & \det(\mathbb{C}^{\{e\}})\otimes H^{\bullet-2}(\mathfrak{M}(C_{k+m}\circ_i C_{l+1})) \\
\xrightarrow{\mathrm{id}\otimes\mathrm{Res}_{(C_{k+1}\circ_j C_{m+1})\circ_i C_{l+1},C_{k+m}\circ_i C_{l+1}}} \quad & \det(\mathbb{C}^{\{e\}})\otimes\det(\mathbb{C}^{\{f\}})\otimes H^{\bullet-3}(\mathfrak{M}((C_{k+1}\circ_j C_{m+1})\circ_i C_{l+1})) \\
\xrightarrow{\mathrm{id}\otimes\phi_{T,C_{k+m}\circ_i C_{l+1},C_{k+l+m-1}}\otimes\mathrm{id}} \quad & \det\mathbb{C}^{\mathrm{In}(T)}\otimes H^{\bullet-3}(\mathfrak{M}(T))
\end{aligned}$$

and

$$\begin{aligned}
& H^{\bullet-1}(\mathfrak{M}_{k+l+m-1}) \\
\xrightarrow{\mathrm{id}\otimes\mathrm{Res}_{C_{k+l}\circ_{j+l-1} C_{m+1},C_{k+l+m-1}}} \quad & \det(\mathbb{C}^{\{f\}})\otimes H^{\bullet-2}(\mathfrak{M}(C_{k+l}\circ_{j+l-1} C_{m+1})) \\
\xrightarrow{\mathrm{id}\otimes\mathrm{Res}_{(C_{k+1}\circ_i C_{l+1})\circ_{j+l-1} C_{m+1},C_{k+l}\circ_{j+l-1} C_{m+1}}} \quad & \det(\mathbb{C}^{\{f\}})\otimes\det(\mathbb{C}^{\{e\}})\otimes H^{\bullet-3}(\mathfrak{M}((C_{k+1}\circ_i C_{l+1})\circ_{j+l-1} C_{m+1})) \\
\xrightarrow{\mathrm{id}\otimes\phi_{T,C_{k+l}\circ_{j+l-1} C_{m+1},C_{k+l+m-1}}\otimes\mathrm{id}} \quad & \det\mathbb{C}^{\mathrm{In}(T)}\otimes H^{\bullet-3}(\mathfrak{M}(T)).
\end{aligned}$$

both coincide with $\mathrm{Res}_{T,C_{k+l+m-1}}$. We are now going to relate the two compositions in (2.3) by expressing them in terms of $\mathrm{Res}_{T,C_{k+l+m-1}}$.

We start with the top-right composition. By Proposition B.22, the diagram

$$\begin{tikzcd}
H^{\bullet-2}(\mathfrak{M}(C_{k+m}\circ_i C_{l+1})) \arrow[r, "\kappa\circ\Psi^*_{C_{k+m},C_{l+1},i}"] \arrow[d, "\mathrm{Res}_{T,C_{k+m}\circ_i C_{l+1}}"'] & H^{\bullet-1}(\mathfrak{M}(C_{k+m}))\otimes H^{\bullet-1}(\mathfrak{M}(C_{l+1})) \arrow[d, "\mathrm{Res}_{C_{k+1}\circ_j C_{m+1},C_{k+m}}\otimes\mathrm{id}"] \\
\det(\mathbb{C}^{\{e\}})\otimes H^{\bullet-3}(\mathfrak{M}(T)) \arrow[r, "\kappa\circ\Psi^*_{C_{k+1}\circ_j C_{m+1},C_{l+1},i}"'] & \det(\mathbb{C}^{\{e\}})\otimes H^{\bullet-2}(\mathfrak{M}(C_{k+1}\circ_j C_{m+1}))\otimes H^{\bullet-1}(\mathfrak{M}(C_{l+1}))
\end{tikzcd}$$

commutes. Thus, after applying appropriate isomorphisms, we can replace the map $\mathrm{Res}_{C_{k+1}\circ_j C_{m+1},C_{k+m}}\otimes\mathrm{id}$ appearing in $\Delta_j\otimes\mathrm{id}$ on the right side of (2.3) with $\mathrm{Res}_{T,C_{k+m}\circ_i C_{l+1}}$.

By Fact B.9 and with the notation thereof, we also have a commutative diagram

$$\begin{tikzcd}
H^{\bullet-3}(\mathfrak{M}(T)) \arrow[r, "\kappa\circ\Psi^*_{C_{k+1}\circ_j C_{m+1}, C_{l+1}, i}"] \arrow[dd, "(\Psi_{C_{k+1}\circ_i C_{l+1}, C_{m+1}, j+l-1}\circ(\Psi_{C_{k+1}, C_{l+1}, i}\times\mathrm{id}))^*"'] & H^{\bullet-2}(\mathfrak{M}(C_{k+1}\circ_j C_{m+1}))\otimes H^{\bullet-1}(\mathfrak{M}_{l+1}) \arrow[d, "(\kappa\circ\Psi^*_{C_{k+1}, C_{m+1}, j})\otimes\mathrm{id}"] \\
 & H^{\bullet-1}(\mathfrak{M}_{k+1})\otimes H^{\bullet-1}(\mathfrak{M}_{m+1})\otimes H^{\bullet-1}(\mathfrak{M}_{l+1}) \arrow[d, "\mathrm{id}\otimes\tau"] \\
H^{\bullet-3}(\mathfrak{M}_{k+1}\times\mathfrak{M}_{l+1}\times\mathfrak{M}_{m+1}) \arrow[r, "\kappa"'] & H^{\bullet-1}(\mathfrak{M}_{k+1})\otimes H^{\bullet-1}(\mathfrak{M}_{l+1})\otimes H^{\bullet-1}(\mathfrak{M}_{m+1})
\end{tikzcd}$$

since

$$\Psi_{C_{k+1}\circ_i C_{l+1}, C_{m+1}, j+l-1}\circ(\Psi_{C_{k+1}, C_{l+1}, i}\times\mathrm{id}) = \Psi_{C_{k+1}\circ_j C_{m+1}, C_{l+1}, i}\circ(\Psi_{C_{k+1}, C_{m+1}, j}\times\mathrm{id})\circ(\mathrm{id}\times\tau).$$

Now let the bijection

$$g\colon \{1,\ldots,k+l+m-2\}\,\dot\cup\,\mathrm{In}(T, C_{k+l+m-1}) \xrightarrow{\cong} \{1,\ldots,k\}\dot\cup\{1,\ldots,l\}\dot\cup\{1,\ldots,m\}$$

be given by

$$g(x) = \begin{cases} i \in \{1,\ldots,k\} & x = e \\ j\in\{1,\ldots,k\} & x = f \\ t\in\{1,\ldots,k\} & x = t\in\{1,\ldots,k+l+m-2\}, t<i \\ t-i+1\in\{1,\ldots,l\} & x = t\in\{1,\ldots,k+l+m-2\}, i\leqslant t< i+l \\ t-l+1\in\{1,\ldots,k\} & x = t\in\{1,\ldots,k+l+m-2\}, i+l\leqslant t<j+l-1 \\ t-j-l+2\in\{1,\ldots,m\} & x = t\in\{1,\ldots,k+l+m-2\}, j+l-1\leqslant t<j+l+m-1 \\ t-l-m+2\in\{1,\ldots,k\} & x = t\in\{1,\ldots,k+l+m-2\}, j+l+m-1\leqslant t \end{cases}.$$

and let

$$\phi_g\colon \det(\mathbb{C}^{\{1,\ldots,k+l+m-2\}})\otimes\det(\mathbb{C}^{\mathrm{In}(T,C_{k+l+m-1})}) \xrightarrow{\cong} \det(\mathbb{C}^{\{1,\ldots,k\}})\otimes\det(\mathbb{C}^{\{1,\ldots,l\}})\otimes\det(\mathbb{C}^{\{1,\ldots,m\}})$$

be the induced isomorphism.

Combining the alternative description of $\mathrm{Res}_{T,C_{k+l+m-1}}$, the previous two diagrams and ϕ_g, we see that the top-right composition of (2.3) is given by

$$\begin{aligned}
&\det(\mathbb{C}^{\{1,\ldots,k+l+m-2\}})\otimes H^{\bullet-1}(\mathfrak{M}_{k+l+m-1}) \\
\xrightarrow{\mathrm{id}\otimes\mathrm{Res}_{T,C_{k+l+m-1}}}\ &\det(\mathbb{C}^{\{1,\ldots,k+l+m-2\}})\otimes\det(\mathbb{C}^{\mathrm{In}(T,C_{k+l+m-1})})\otimes H^{\bullet-3}(\mathfrak{M}(T)) \\
\xrightarrow{\mathrm{id}\otimes\Psi^*}\ &\det(\mathbb{C}^{\{1,\ldots,k+l+m-2\}})\otimes\det(\mathbb{C}^{\mathrm{In}(T,C_{k+l+m-1})})\otimes H^{\bullet-3}(\mathfrak{M}_{k+1}\times\mathfrak{M}_{l+1}\times\mathfrak{M}_{m+1}) \\
\xrightarrow{\mathrm{id}\otimes\kappa}\ &\det(\mathbb{C}^{\{1,\ldots,k+l+m-2\}})\otimes\det(\mathbb{C}^{\mathrm{In}(T,C_{k+l+m-1})})\otimes H^{\bullet-1}(\mathfrak{M}_{k+1})\otimes H^{\bullet-1}(\mathfrak{M}_{l+1})\otimes H^{\bullet-1}(\mathfrak{M}_{m+1}) \\
\xrightarrow{\phi_g\otimes\mathrm{id}}\ &\det(\mathbb{C}^{\{1,\ldots,k\}})\otimes\det(\mathbb{C}^{\{1,\ldots,l\}})\otimes\det(\mathbb{C}^{\{1,\ldots,m\}})\otimes H^{\bullet-1}(\mathfrak{M}_{k+1})\otimes H^{\bullet-1}(\mathfrak{M}_{l+1})\otimes H^{\bullet-1}(\mathfrak{M}_{m+1}) \\
\xrightarrow{\sigma}\ &\det(\mathbb{C}^{\{1,\ldots,k\}})\otimes H^{\bullet-1}(\mathfrak{M}_{k+1})\otimes\det(\mathbb{C}^{\{1,\ldots,l\}})\otimes H^{\bullet-1}(\mathfrak{M}_{l+1})\otimes\det(\mathbb{C}^{\{1,\ldots,m\}})\otimes H^{\bullet-1}(\mathfrak{M}_{m+1})
\end{aligned}\tag{2.4}$$

where $\Psi = (\Psi_{C_{k+1},C_{l+1},i}\times\mathrm{id})\circ\Psi_{C_{k+1}\circ_i C_{l+1},C_{m+1},j+l-1}$ and σ is the appropriate permutation of the tensor factors. A similar analysis of the left-bottom composition of (2.3) shows that it is also given by (2.4), thus showing that (2.3) indeed commutes.

Now let $j'\in\{1,\ldots,l\}$. The sequential cocomposition axiom requires that the diagram

$$\begin{tikzcd}
\mathscr{CoGrav}(k+l+m-2) \arrow[r, "\Delta_i"] \arrow[d, "\Delta_{i+j'-1}"'] & \mathscr{CoGrav}(k) \otimes \mathscr{CoGrav}(l+m-1) \arrow[d, "\mathrm{id}\otimes\Delta_{j'}"] \\
\mathscr{CoGrav}(k+l-1) \otimes \mathscr{CoGrav}(m) \arrow[r, "\Delta_i\otimes\mathrm{id}"'] & \mathscr{CoGrav}(k) \otimes \mathscr{CoGrav}(l) \otimes \mathscr{CoGrav}(m)
\end{tikzcd}$$

commutes. As for (2.3), the two compositions in this diagram can be described in terms of a residue map associated to contracting two edges. We omit the details of this analysis as it is completely analogous. $\square$

We can now define the gravity operad.

Definition 2.4. The *gravity operad* $\mathscr{Grav}$ is the operad in the category of graded $\mathbb{C}$-vector spaces given by the linear dual of $\mathscr{CoGrav}$, i. e. its underlying Σ-module is given by

$$\mathscr{Grav}(n) := (\mathscr{CoGrav}(n))^* = (\det(\mathbb{C}^n) \otimes H^{\bullet-1}(\mathfrak{M}_{n+1}))^* \cong \det(\mathbb{C}^n) \otimes H_{\bullet-1}(\mathfrak{M}_{n+1})$$

and its infinitesimal composition morphisms $\circ_i^{\mathscr{Grav}}$ by

$$\begin{aligned} \mathscr{Grav}(k) \otimes \mathscr{Grav}(l) &\cong (\mathscr{CoGrav}(k) \otimes \mathscr{CoGrav}(l))^* \\ \xrightarrow{\Delta_i^*} \qquad & \mathscr{CoGrav}(k+l-1)^* = \mathscr{Grav}(k+l-1). \end{aligned}$$

2.2 Koszul Duality

In this subsection, we are going the discuss the Koszul duality between the gravity operad and the hypercommutative operad which was proven in [GK94] and in [Get95].

We start by defining the hypercommutative operad which plays an important role in mathematical physics because it describes the algebraic structure of quantum cohomology of varieties (cf. [KM94]).

Definition 2.5. The *hypercommutative operad* $\mathscr{HyCom}$ is given by the homology of the operad $(\overline{\mathfrak{M}}_{n+1})_{n\in\mathbb{N}_+}$ of Fact B.10, i. e. its underlying Σ-module is given by

$$\mathscr{HyCom}(n) := H_\bullet(\overline{\mathfrak{M}}_{n+1})$$

and its infinitesimal composition morphisms by

$$\circ_i^{\mathscr{HyCom}} \colon H_\bullet(\overline{\mathfrak{M}}_{k+1}) \otimes H_\bullet(\overline{\mathfrak{M}}_{l+1}) \xrightarrow{\kappa} H_\bullet(\overline{\mathfrak{M}}_{k+1} \times \overline{\mathfrak{M}}_{l+1}) \xrightarrow{H_\bullet(\circ_i)} H_\bullet(\overline{\mathfrak{M}}_{k+l}).$$

Next, we want to analyze the cobar construction $\Omega\mathscr{Grav}$ of $\mathscr{Grav}$.

Remark 2.6. For $n \in \mathbb{N}$ consider the double complex $(K^{p,q}_{\mathscr{Grav},n})_{p,q\in\mathbb{N}}$ underlying $(\Omega\mathscr{Grav})(n)$. Then, for $k \in \mathbb{N}$ we have

$$K^{k+1,\bullet}_{\mathscr{Grav},n} = \det(\mathbb{C}^n) \otimes \left(\bigoplus_{\substack{T\in\mathscr{T}_{n+1} \\ |\operatorname{In}(T)|=k}} \det(\mathbb{C}^{\operatorname{In}(T)}) \otimes \bigotimes_{v\in\operatorname{NLV}(T)} \mathscr{Grav}(|\operatorname{Adj}(v)|-1)^* \right) \tag{2.5}$$

where the internal grading $\bullet$ on the left hand side comes from the grading of $\bigotimes_{v\in \mathrm{NLV}(T)} \mathcal{G}\mathit{rav}(|\operatorname{Adj}(v)|-1)^*$.

Here the Σ_n-action on the first tensor factor is given by permuting the coordinates of $\mathbb{C}^n$. The action on the second factor is given by acting on trees, i. e. $\sigma \in \Sigma_n$ maps the summand indexed by T to the summand indexed by $\sigma(T)$ as follows: Since T and $\sigma(T)$ have the same underlying unlabeled tree, we can identify $\mathrm{In}(T)$ with $\mathrm{In}(\sigma(T))$ to obtain an isomorphism $\det(\mathbb{C}^{\mathrm{In}(T)}) \cong \det(\mathbb{C}^{\mathrm{In}(\sigma(T))})$. Similarly, $\mathrm{NLV}(T)$ can be identified with $\mathrm{NLV}(\sigma(T))$, which yields an isomorphism

$$\bigotimes_{v\in \mathrm{NLV}(T)} \mathcal{G}\mathit{rav}(|\operatorname{Adj}(v)|-1|)^* \cong \bigotimes_{v'\in \mathrm{NLV}(\sigma(T))} \mathcal{G}\mathit{rav}(|\operatorname{Adj}(v')|-1)^*$$

by mapping the factor indexed by a vertex in $\mathrm{NLV}(T)$ to the factor indexed by the corresponding vertex in $\mathrm{NLV}(\sigma(T))$.

By identifying $\mathcal{G}\mathit{rav}^*$ with $\mathcal{C}\mathit{o}\mathcal{G}\mathit{rav}$ and using the definition of $\mathcal{C}\mathit{o}\mathcal{G}\mathit{rav}$, we can identify $K^{k+1,\bullet}_{\mathcal{G}\mathit{rav},n}$ with

$$\det(\mathbb{C}^n)\otimes\left(\bigoplus_{\substack{T\in\mathcal{T}_{n+1}\\ |\mathrm{In}(T)|=k}} \det(\mathbb{C}^{\mathrm{In}(T)})\otimes \bigotimes_{v\in\mathrm{NLV}(T)}\left(\det(\mathbb{C}^{|\operatorname{Adj}(v)|-1})\otimes H^{\bullet-1}(\mathfrak{M}_{|\operatorname{Adj}(v)|})\right)\right). \tag{2.6}$$

Note that for every tree $T\in\mathcal{T}_{n+1}$, identifying the labels $\{1,\dots,n+1\}$ with the corresponding external edges $\{e_1,\dots,e_{n+1}\}$ yields a bijection

$$\{1,\dots,n\}\,\dot\cup\,\mathrm{In}(T)\cong \mathrm{Edge}(T)\backslash\{e_{n+1}\}.$$

Now identifying $\{1,\dots,|\operatorname{Adj}(v)|-1\}$ with the first $|\operatorname{Adj}(v)|-1$ edges of the corolla $C_{|\operatorname{Adj}(v)|}$ in the corresponding grafting while decomposing the underlying tree of T into corollas yields another bijection

$$\mathrm{Edge}(T)\backslash\{e_{n+1}\}\cong \dot\bigcup_{\mathrm{NLV}(T)}\{1,\dots,|\operatorname{Adj}(v)|-1\}.$$

Moreover, these bijections are compatible with the action of Σ_n. Thus, the determinants in (2.6) can be trivialized to identify the Σ_n-representation $K^{k+1,\bullet}_{\mathcal{G}\mathit{rav},n}$ with

$$\bigoplus_{\substack{T\in\mathcal{T}_{n+1}\\ |\mathrm{In}(T)|=k}}\ \bigotimes_{v\in\mathrm{NLV}(T)} H^{\bullet-1}(\mathfrak{M}_{|\operatorname{Adj}(v)|}). \tag{2.7}$$

Now note that for every tree T, we have $|\mathrm{NLV}(T)|=|\mathrm{In}(T)|+1$. Thus, using the Künneth isomorphism and the product decomposition of Fact B.9, we can further identify this with

$$\bigoplus_{\substack{T\in\mathcal{T}_{n+1}\\ |\mathrm{In}(T)|=k}} H^{\bullet-1-k}(\mathfrak{M}(T)). \tag{2.8}$$

Next, we are going to identify the (external) differential $d^{k+1}\colon K^{k+1,\bullet}_{\mathcal{G}rav,n} \to K^{k+2,\bullet}_{\mathcal{G}rav,n}$. Let $T, T' \in \mathcal{T}_{n+1}$ such that $|\operatorname{In}(T)| = k$, $|\operatorname{In}(T')| = k+1$ and $T \cong T'/e$. By

$$\lambda_{T',T}\colon \det(\mathbb{C}^{\operatorname{In}(T)}) \to \det(\mathbb{C}^{\operatorname{In}(T')})$$

we denote the map given by wedging with the basis element of $\mathbb{C}^{\operatorname{In}(T')}$ which corresponds to $e \in \operatorname{In}(T')$.

Now let $v_1, v_2 \in \operatorname{NLV}(T')$ be the vertices adjacent to $e \in \operatorname{In}(T')$ and $v_0 \in \operatorname{NLV}(T)$ be the vertex they collapse into. Then we can identify $\operatorname{NLV}(T)\backslash\{v_0\}$ with $\operatorname{NLV}(T')\backslash\{v_1, v_2\}$. Thus, denoting both of these sets by V, we obtain identifications

$$\begin{aligned}&\bigotimes_{v \in \operatorname{NLV}(T)} \mathcal{G}rav(|\operatorname{Adj}(v)| - 1|)^* \\ &\cong \mathcal{G}rav(|\operatorname{Adj}(v_0)| - 1)^* \otimes \bigotimes_{v \in V} \mathcal{G}rav(|\operatorname{Adj}(v)| - 1|)^*\end{aligned}$$

and

$$\begin{aligned}&\bigotimes_{v \in \operatorname{NLV}(T')} \mathcal{G}rav(|\operatorname{Adj}(v)| - 1|)^* \\ &\cong \mathcal{G}rav(|\operatorname{Adj}(v_1)| - 1)^* \otimes \mathcal{G}rav(|\operatorname{Adj}(v_2)| - 1)^* \otimes \bigotimes_{v \in V} \mathcal{G}rav(|\operatorname{Adj}(v)| - 1|)^*.\end{aligned}$$

Moreover, by building T and T' by grafting corollas, the edge contraction $T' \geqslant_1 T$ can be identified with an edge contraction $C_{|\operatorname{Adj}(v_1)|} \circ_i C_{|\operatorname{Adj}(v_2)|} \geqslant_1 C_{|\operatorname{Adj}(v_0)|}$ at some stage. Thus, we obtain a map

$$\gamma_{T',T}\colon \bigotimes_{v \in \operatorname{NLV}(T)} \mathcal{G}rav(|\operatorname{Adj}(v)| - 1|)^* \to \bigotimes_{v' \in \operatorname{NLV}(T')} \mathcal{G}rav(|\operatorname{Adj}(v')| - 1|)^*$$

which under the above identifications corresponds to $(\circ_i^{\mathcal{G}rav})^* \otimes \mathrm{id}$.

Now, under these notations and identifications, the differential d^{k+1} is given by adding up the maps $\mathrm{id}_{\det(\mathbb{C}^n)} \otimes \lambda_{T',T} \otimes \gamma_{T',T}$ for all 1-step edge contractions $T' \geqslant_1 T$ with $\operatorname{In}(T) = k$.

When we identify $\mathcal{G}rav^*$ with $\mathcal{C}o\mathcal{G}rav$ as in (2.6), $(\circ_i^{\mathcal{G}rav})^*$ gets identified with Δ_i. Trivializing the determinants in (2.7) gets rid of the factors $\mathrm{id}_{\det(\mathbb{C}^n)} \otimes \lambda_{T',T}$ in each summand of the differential. Finally, identifying $\bigotimes_{v \in \operatorname{NLV}(T)} H^{\bullet-1}(\mathfrak{M}_{|\operatorname{Adj}(v)|})$ with $H^{\bullet-1-|\operatorname{In}(T)|}(\mathfrak{M}(T))$ and utilizing the compatibility of residue morphisms with product decompositions as in Proposition B.22, we see that in the description of (2.8) the summand of the differential corresponding to an edge contraction $T' \geqslant_1 T$ with $T \cong T'/e$ is given by

$$\operatorname{Res}^{(e)}\colon H^{\bullet-1-|\operatorname{In} T|}(\mathfrak{M}(T)) \to H^{\bullet-1-|\operatorname{In} T'|}(\mathfrak{M}(T')).$$

Let us now also identify the composition morphisms of $\Omega\mathcal{G}rav$ which are inherited from the composition morphisms of the free operad $\mathfrak{F}(\mathcal{G}rav^*[-1])$. Let $T \in \mathcal{T}_{l+1}$, $T' \in \mathcal{T}_{m+1}$, $i \in \{1, \ldots, l\}$. Note that we can identify $\operatorname{NLV}(T \circ_i$

T') with $\mathrm{NLV}(T) \dot\cup \mathrm{NLV}(T')$. Now, after trivializing the determinants occurring in $\Omega\mathcal{G}rav$ as in (2.7), the composition morphism $\circ_i^{\Omega\mathcal{G}rav}$ on the summand

$$\left(\bigotimes_{v\in\mathrm{NLV}(T)} H^{\bullet-1}(\mathfrak{M}_{|\operatorname{Adj}(v)|})\right) \otimes \left(\bigotimes_{v'\in\mathrm{NLV}(T')} H^{\bullet-1}(\mathfrak{M}_{|\operatorname{Adj}(v')|})\right)$$

of $K^{|\operatorname{In}(T)|+1,\bullet}_{\mathcal{G}rav,l} \otimes K^{|\operatorname{In}(T')|+1,\bullet}_{\mathcal{G}rav,m}$ is given by mapping the factor indexed by $v \in \mathrm{NLV}(T) \dot\cup \mathrm{NLV}(T')$ to the factor indexed by the corresponding vertex in $\mathrm{NLV}(T \circ_i T')$ in the summand

$$\bigotimes_{v\in\mathrm{NLV}(T\circ_i T')} H^{\bullet-1}(\mathfrak{M}_{|\operatorname{Adj}(v)|})$$

of $K^{|\operatorname{In}(T)|+1+|\operatorname{In}(T')|+1,\bullet}_{\mathcal{G}rav,l+m-1} = K^{|\operatorname{In}(T\circ_i T')|+1,\bullet}_{\mathcal{G}rav,l+m-1}$. After utilizing Künneth isomorphisms and product decomposition of $\mathfrak{M}(T)$, $\mathfrak{M}(T')$ resp. $\mathfrak{M}(T \circ_i T')$, we see that in the description of (2.8), the composition morphism $\circ_i^{\Omega\mathcal{G}rav}$ is given by adding up the isomorphisms

$$\begin{aligned}
& H^{\bullet-1-|\operatorname{In}(T)|}(\mathfrak{M}(T)) \otimes H^{\bullet-1-|\operatorname{In}(T')|}(\mathfrak{M}(T')) \\
\xrightarrow{\kappa}\quad & H^{\bullet-2-|\operatorname{In}(T)|-|\operatorname{In}(T')|}(\mathfrak{M}(T) \times \mathfrak{M}(T')) \\
\xrightarrow{(\Psi^{-1}_{T,T',i})^*}\quad & H^{\bullet-1-|\operatorname{In}(T\circ_i T')|}(\mathfrak{M}(T \circ_i T')).
\end{aligned}$$

Now that we have an explicit description of $\Omega\mathcal{G}rav$, we can show that it is quasi-isomorphic to $\mathcal{H}y\mathcal{C}om$ by using the residue spectral sequence.

Theorem 2.7. *There is an quasi-isomorphism $\Omega\mathcal{G}rav \simeq \mathcal{H}y\mathcal{C}om$ of differential graded operads.*

Proof. Let $(E^{p,q})_{p,q\in\mathbb{N}}$ be the first page of the residue spectral sequence for $\overline{\mathfrak{M}}_{n+1}$ (cf. Fact B.24) which we consider as a double complex with zero vertical differentials. Our previous analysis yields isomorphisms $K^{p+1,q}_{\mathcal{G}rav,n} \cong E^{p,q-1}$ for $p \in \mathbb{N}$, $q \in \mathbb{N}_+$ which are compatible with the differentials. Considering that $K^{\bullet,0}_{\mathcal{G}rav,n} \cong K^{0,\bullet}_{\mathcal{G}rav,n} \cong 0$, this yields an isomorphism

$$\operatorname{tot}((K^{p,q}_{\mathcal{G}rav,n})_{p,q\in\mathbb{N}})^\bullet \cong \operatorname{tot}((E^{p,q})_{p,q\in\mathbb{N}})^\bullet$$

of total complexes. Thus, using the exact sequences of Fact B.25 we obtain a quasi-isomorphism

$$(\Omega\mathcal{G}rav)(n) = \operatorname{tot}((K^{p,q}_{\mathcal{G}rav,n})_{p,q\in\mathbb{N}})^{n-2-\bullet} \cong \operatorname{tot}((E^{p,q})_{p,q\in\mathbb{N}})^{n-2-\bullet} \simeq H^{n-2-\bullet}(\overline{\mathfrak{M}}_{n+1}).$$

Now using Poincaré duality and considering all $n \in \mathbb{N}_+$, we get a quasi-isomorphism $\Omega\mathcal{G}rav \simeq \mathcal{H}y\mathcal{C}om$ of Σ-modules. Since the composition morphisms of $\Omega\mathcal{G}rav$ and $\mathcal{H}y\mathcal{C}om$ are both given by combining Künneth isomorphisms with $\Psi_{T,S,i}$ for suitable T, S and i, this quasi-isomorphism is compatible with the operad structures on both sides and hence yields a quasi-isomorphism of differential graded operads. □

This result can be seen as an extension of the classical Koszul duality between commutative algebras and Lie algebras since the suboperad of $\mathcal{HyCom}$ generated by $\mathcal{HyCom}(2)$ is isomorphic to the commutative operad and the suboperad of $\mathcal{Grav}$ generated by $\mathcal{Grav}(2)$ is isomorphic to the Lie operad (cf. [Get95]).

A Labeled Trees

In this appendix we fix some notations regarding labeled trees which play a crucial role while describing operadic compositions and stratifications of moduli spaces of genus 0 curves. A detailed account of trees and their relation to operads can be found in [BM08, Section 1] and [LV12, Appendix C].

Definition A.1. A *tree* is a connected undirected graph with no cycles.

Now let T be a tree.

Notation A.2.
- By $\mathrm{Vert}(T)$ we denote the set of vertices of T.
- For $v \in \mathrm{Vert}(T)$, we denote by $\mathrm{Adj}(v)$ the set of edges which are adjacent to v.
- Let $\mathrm{NLV}(T)$ denote the set of those vertices of T which are not leaves, i. e. $\mathrm{NLV}(T) = \{v \in V \mid \mathrm{Adj}(v) > 1\}$.
- By $\mathrm{Edge}(T)$ we denote the set of edges of T.

Definition A.3.
- An edge $e \in \mathrm{Edge}(T)$ is called *internal* if all vertices which are adjacent to e are also adjacent to another edge. Let $\mathrm{In}(T)$ denote the set of internal edges of T.
- An edge $e \in \mathrm{Edge}(T)$ is called *external* if it is not internal, i. e. if one of the vertices adjacent to e is not adjacent to any other edge. Let $\mathrm{Ex}(T)$ denote the set of internal edges of T.

Definition A.4. For $e \in \mathrm{Edge}(T)$ let T/e denote the tree obtained from T by removing the edge e and identifying the two vertices adjacent to e. We say that T/e is *obtained from T by contracting the edge e*.

Definition A.5. Let $n \in \mathbb{N}_+$. An *n-tree* is a tree whose external edges are labeled with $1, \ldots, n$, i. e. a tree S together with an identification $\mathrm{Ex}(S) \cong \{1, \ldots, n\}$. Let $\mathcal{T}_n$ denote the set of isomorphism classes of n-trees w. r. t. graph isomorphisms which respect the labels of the external edges.

Remark A.6. The symmetric group Σ_n acts on the class of n-trees by permuting the labels. Moreover, this action is compatible with graph isomorphisms which preserve labels, so it descends to an action of Σ_n on $\mathcal{T}_n$.

We will sometimes identify an n-tree with its isomorphism class or its underlying tree, but the object we are talking about will be clear from the context.

From now on we fix an $n \in \mathbb{N}_+$ and $T, S \in \mathcal{T}_n$.

Remark A.7. For an internal edge $e \in \mathrm{In}(T)$, the external edges of T/e can be identified with those of T and thus inherit the labels of $\mathrm{Ex}(T)$. When we consider T/e as an n-tree, we do it so using these inherited labels.

Notation A.8.
- We write $S \geqslant_1 T$ if T is isomorphic to S/e as an n-tree for some $e \in \mathrm{In}(S)$.
- We write $S \geqslant T$ if there is a sequence $T_1, \ldots, T_k$ of n-trees with $k \in \mathbb{N}_+$ and such that $T_1 = S$, $T_k = T$ and $T_i \geqslant_1 T_{i+1}$ for $i \in \{1, \ldots, k-1\}$, i. e. if T is obtained from S via a sequence of internal edge contractions.
- For $S \geqslant T$ we denote by $\mathrm{In}(S,T) \subseteq \mathrm{In}(S)$ the set of those internal edges which are contracted to obtain T.

When we think of the $(n+1)$-st external edge of an $(n+1)$-tree as an "output" and the remaining external edges as "inputs", we can regard an $(n+1)$-tree as a "composition scheme with n inputs". We now want to describe how these composition schemes can be "grafted". For this we let $n, m \in \mathbb{N}_+$, $T \in \mathcal{T}_{n+1}$, $S \in \mathcal{T}_{m+1}$ and $i \in \{1, \ldots, n\}$.

Definition A.9. We define an $(n+m)$-tree $T \circ_i S$ as follows:

Let e_i be the i-th external edge of T which connects the vertices $v_T, w_T \in \mathrm{Vert}(T)$, where w_T is only adjacent to e_i. Let f_{m+1} be the $(m+1)$-st external edge of S which connects the vertices $v_S, w_S \in \mathrm{Vert}(S)$, where w_S is only adjacent to f_{m+1}. The underlying tree R of $T \circ_i S$ is the tree obtained from the disjoint union of T and S by removing the edges e_i and f_{m+1} along with the vertices w_T and w_T, and replacing it with an (internal) edge connecting v_T and v_S.

Thus, if we fix enumerations $\mathrm{Ex}(T) = \{e_1, \ldots, e_{n+1}\}$ and $\mathrm{Ex}(S) = \{f_1, \ldots, f_{m+1}\}$ given by the labels, we obtain

$$\mathrm{Ex}(R) = (\mathrm{Ex}(T) \dot{\cup} \mathrm{Ex}(S)) \backslash \{e_i, f_{m+1}\}.$$

Now we enumerate these external edges as

$$e_1, \ldots, e_{i-1}, f_1, \ldots, f_m, e_{i+1}, \ldots, e_{n+1}$$

which yields an identification of $\mathrm{Ex}(R)$ with $\{1, \ldots, n+m\}$. We now define $T \circ_i S$ to be R equipped with this identification.

In fact, one can build every tree from simple pieces via grafting:

Definition A.10. We define the *n-corolla* C_n to be the unique n-tree (isomorphism class) with no internal edges.

Remark A.11. Every labeled tree can be realized via a sequence of graftings of corolla up to a permutation of its labels, i. e. for each tree $R \in \mathcal{T}_n$ there are sequences $(n_0, \ldots, n_k)$, $(i_1, \ldots, i_k)$ and a permutation $\sigma \in \Sigma_n$ such that R is isomorphic to $\sigma(C_{n_0} \circ_{i_1} C_{n_1} \circ_{i_2} \cdots \circ_{i_k} C_{n_k})$ as a labeled tree.

Moreover, grafting is compatible with certain permutations of labels.

Notation A.12.

- Let $\theta_{n,i}\colon \Sigma_n \hookrightarrow \Sigma_{n+m-1}$ be given by letting $\sigma \in \Sigma_n$ act on $\{1,\ldots,n+m-1\}$ by considering the block $\{i,\ldots,i+m-1\}$ as one element and permuting the resulting $(i-1)+1+(n-i)=n$ elements.
- Let $\vartheta_{m,i}\colon \Sigma_m \hookrightarrow \Sigma_{n+m-1}$ be defined by letting $\rho \in \Sigma_m$ act on the block $\{i,\ldots,i+m-1\} \subseteq \{1,\ldots,n+m-1\}$.

Remark A.13. We endow $\mathcal{T}_{n+1} \times \mathcal{T}_{m+1}$ with a Σ_n-action by permuting the first n labels in the first coordinate and a Σ_m-action by permuting the first m labels in the second coordinate. We endow $\mathcal{T}_{n+m}$ with a Σ_n- resp. Σ_m action by acting on the first $n+m-1$ coordinates via $\theta_{n,i}$ resp. $\vartheta_{m,i}$.

With respect to these actions, the map

$$\circ_i\colon \mathcal{T}_{n+1} \times \mathcal{T}_{m+1} \to \mathcal{T}_{n+m}$$

is Σ_n- and Σ_m-equivariant.

B Moduli Spaces of Marked Genus 0 Curves

In this appendix we review some facts related to moduli spaces of genus 0 curves which are used throughout Section 2 while dealing with the gravity operad. Many of these results were originally proven in the context of algebraic geometry, but with the exception the terminology "curve" (instead of "Riemannian surface"), we use the language of complex geometry which is a better fit for our considerations of residue morphisms.

B.1 Moduli Spaces

We start by defining the geometric objects we are concerned with. For this, we fix a natural number $n \geqslant 3$.

Definition B.1. The *moduli space of smooth genus* 0 *curves with* n *marked points* is

$$\mathfrak{M}_n := \left\{(p_1, \dots, p_n) \in (\mathbb{C}P^1)^n \mid p_i \neq p_j \text{ for } i \neq j\right\} \big/ \mathrm{PGL}_2(\mathbb{C}),$$

where $\mathrm{PGL}_2(\mathbb{C})$ acts diagonally.

Remark B.2. Since the action of $\mathrm{PGL}_2(\mathbb{C})$ on $\mathbb{C}P^1$ is strictly 3-transitive, moving the last three points to 0, 1 and ∞ yields an isomorphism

$$\mathfrak{M}_n \cong \{(z_1, \dots, z_{n-3}) \in (\mathbb{C} \setminus \{0,1\})^n \mid z_i \neq z_j \text{ for } i \neq j\}.$$

In particular, $\mathfrak{M}_n$ is an $(n-3)$-dimensional affine variety.

The following convention helps us deal with some degenerate cases.

Convention B.3. We set $\mathfrak{M}_2 := \varnothing$.

Remark B.4. Σ_n acts on $\mathfrak{M}_n$ by permuting the marked points. Moreover, there is a unique action of Σ_2 on $\mathfrak{M}_2 = \varnothing$ (which is necessarily trivial).

Next, we want to describe a compactification of $\mathfrak{M}_n$ which goes back to [Knu83a] and [Knu83b]. Another approach to this compactification via a sequence of blow-ups can be found in [Kee92]. In the following we will need some concepts related to trees which are dealt with in Appendix A.

Fact B.5 ([Knu83a, Theorem 2.7], [Knu83b, Theorem 6.1])**.** *$\mathfrak{M}_n$ admits a compactification $\overline{\mathfrak{M}}_n$ which classifies n-marked stable curves of genus* 0*. $\overline{\mathfrak{M}}_n$ has the following properties:*

- *$\overline{\mathfrak{M}}_n$ is a smooth projective variety.*
- *$\overline{\mathfrak{M}}_n$ admits a stratification*

$$\overline{\mathfrak{M}}_n = \dot{\bigcup_{T \in \mathcal{T}_n}} \mathfrak{M}(T) \tag{B.1}$$

given by dual graphs of stable curves such that $\overline{\mathfrak{M}(T')} \subseteq \overline{\mathfrak{M}(T)}$ if and only if $T' \geqslant T$.

- *$\mathfrak{M}_n$ can be identified with the stratum $\mathfrak{M}(C_n)$.*
- *For $T \in \mathcal{T}_n$, $\mathfrak{M}(T)$ has codimension $|\operatorname{In}(T)|$ in $\overline{\mathfrak{M}}_n$.*

Notation B.6. For an n-tree T let $\overline{\mathfrak{M}}(T)$ denote the closure $\overline{\mathfrak{M}(T)}$ of the stratum $\mathfrak{M}(T) \subseteq \overline{\mathfrak{M}}_n$ in $\overline{\mathfrak{M}}_n$.

Convention B.7. We set $\overline{\mathfrak{M}}_2 := \varnothing$.

Remark B.8. The Σ_n-action on $\mathfrak{M}_n$ extends to $\overline{\mathfrak{M}}_n$ in a way that is compatible with the stratification of Fact B.5, i. e. maps strata into strata. The induced action on the set of strata coincides with the one induced by the Σ_n-action on $\mathcal{T}_n$ given by permuting the labels of external edges (cf. Remark A.6).

Similarly, the (trivial) Σ_2-action on $\mathfrak{M}_2$ induces a (trivial) Σ_2-action $\overline{\mathfrak{M}}_2 = \varnothing$.

One can in fact describe the strata in (B.1) more explicitly.

Fact B.9 ([Knu83a, Theorem 3.7])**.** *Let $T \in \mathcal{T}_{k+1}$, $S \in \mathcal{T}_{l+1}$, $T' \succcurlyeq T$ and $S' \succcurlyeq S$.*

Then grafting of stable curves induces an isomorphism

$$\Psi_{T,S,i} \colon \overline{\mathfrak{M}}(T) \times \overline{\mathfrak{M}}(S) \xrightarrow{\cong} \overline{\mathfrak{M}}(T \circ_i S)$$

which is compatible with the stratification in (B.1) in the sense that it restricts to an isomorphism between $\mathfrak{M}(T') \times \mathfrak{M}(S')$ and $\mathfrak{M}(T' \circ_i S')$. By abuse of notation, we will denote the restriction of $\Psi_{T,S,i}$ to $\mathfrak{M}(T) \times \mathfrak{M}(S)$ also by $\Psi_{T,S,i}$.

Moreover, the restriction of $\Psi_{T,S,i}$ to $\overline{\mathfrak{M}}(T') \times \overline{\mathfrak{M}}(S')$ coincides with $\Psi_{T',S',i}$.

In particular, for every tree $T \in \mathcal{T}_n$, decomposing a suitable relabeling of T into corollas by representing each internal edge as a grafting (cf. Remark A.11) yields isomorphisms

$$\mathfrak{M}(T) \cong \prod_{v \in \mathrm{NLV}(T)} \mathfrak{M}_{|\operatorname{Adj}(v)|}.$$

and

$$\overline{\mathfrak{M}}(T) \cong \prod_{v \in \mathrm{NLV}(T)} \overline{\mathfrak{M}}_{|\operatorname{Adj}(v)|}.$$

Different orderings of $\operatorname{In}(T)$ yield product decompositions which differ from each other by a permutation of their factors.

In fact, these product decompositions enjoy certain compatibility properties with the action of the symmetric group which can be compactly formulated as an operad structure.

Fact B.10 ([GK94, 1.4]). *The isomorphisms $(\Psi_{T,S,i})_{T,S,i}$ of Fact B.9 can be chosen in a way that the Σ-module $(\overline{\mathfrak{M}}_{n+1})_{n\in\mathbb{N}_+}$ in the category of varieties over $\mathbb{C}$, where Σ_n acts on $\overline{\mathfrak{M}}_{n+1}$ by permuting the first n coordinates, can be endowed with the structure of an operad via the infinitesimal compositions maps*

$$\circ_i\colon \overline{\mathfrak{M}}_{k+1} \times \overline{\mathfrak{M}}_{l+1} \xrightarrow{\Psi_{C_{k+1},C_{l+1},i}} \overline{\mathfrak{M}}(C_{k+1} \circ_i C_{l+1}) \subseteq \overline{\mathfrak{M}}_{k+l+1}$$

for $k, l \in \mathbb{N}_+$, $1 \leqslant i \leqslant k$.

One can also describe the local picture of the stratification which will turn out to be given by the following standard stratification.

Construction B.11. Let $k, l \in \mathbb{N}$ with $l \leqslant k$ and let $D := \{(z_1, \ldots, z_k) \in \mathbb{C}^k \mid z_1 \cdot \ldots \cdot z_l = 0\}$. This datum induces the *standard stratification of $\mathbb{C}^k$ with boundary D* given by

$$\dot{\bigcup_{I\subseteq\{1,\ldots,l\}}} X(I)$$

where

$$X(I) := \left\{(z_1, \ldots, z_k) \in \mathbb{C}^k \mid z_i = 0 \text{ for } i \in I, z_j \neq 0 \text{ for } j \in \{1, \ldots, l\}\backslash I\right\}$$

with

$$\overline{X(I)} = \left\{(z_1, \ldots, z_k) \in \mathbb{C}^k \mid z_i = 0 \text{ for } i \in I\right\}$$

and $\overline{X(I')} \subseteq \overline{X(I)}$ if and only if $I' \supseteq I$.

Fact B.12 ([Knu83a, Theorem 2.7]). *Let $T \in \mathcal{T}_n$, $x \in \overline{\mathfrak{M}}(T)$. Let $S \in \mathcal{T}_n$ be the unique n-tree such that $x \in \mathfrak{M}(S)$. Then $S \geqslant T$, so T is obtained from S by contracting some edges $e_1, \ldots, e_l$. For $I \subseteq \{1, \ldots, l\}$ let T_I the n-tree obtained from S by contracting the edges e_i with $i \in \{1, \ldots, l\}\backslash I$.*

Then there is a chart of $\overline{\mathfrak{M}}(T)$ centered at x such that on that chart, the stratification of $\overline{\mathfrak{M}}(T)$ induced by (B.1) coincides with the standard stratification of Construction B.11 in the sense that $\mathfrak{M}(T_I)$ corresponds to $X(I)$.

In particular, $\partial\overline{\mathfrak{M}}(T) := \overline{\mathfrak{M}}(T)\backslash\mathfrak{M}(T)$ is a normal crossing divisor in $\overline{\mathfrak{M}}(T)$ with irreducible components $\{\overline{\mathfrak{M}}(T') \mid T' \geqslant_1 T\}$.

B.2 Logarithmic Forms and Residue Morphisms

Next, we fix an n-tree $T \in \mathcal{T}_n$ and want to describe differential forms on $\overline{\mathfrak{M}}(T)$ with logarithmic poles along $\partial\overline{\mathfrak{M}}(T)$ resp. corresponding residue maps which play a crucial role in the construction of the gravity operad. A more detailed treatment of these concepts for general normal crossing divisors with smooth components can be found in [PS08, Chapter 4].

Definition B.13. We define a complex of sheaves on $\overline{\mathfrak{M}}(T)$, the *logarithmic de Rham complex*

$$\Omega^{\bullet}_{\overline{\mathfrak{M}}(T)}(\log \partial\overline{\mathfrak{M}}(T))$$

of $(\overline{\mathfrak{M}}(T), \partial(\overline{\mathfrak{M}}(T))$ by describing its sections on charts as in Fact B.12 where the stratification is given by the standard stratification of Construction B.11.

Note that in such charts, the boundary $\partial\overline{\mathfrak{M}}(T)$ corresponds to $\{(z_1, \ldots, z_k) \in \mathbb{C}^k \mid z_1 \cdot \cdots \cdot z_l = 0\}$. Now the sections of $\Omega^{\bullet}_{\overline{\mathfrak{M}}(T)}(\log \partial\overline{\mathfrak{M}}(T))$ on such a chart are generated by forms of the form

$$\frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_m}}{z_{i_m}} \wedge \eta \tag{B.2}$$

where $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots l\}$ and η is a holomorphic form on $\overline{\mathfrak{M}}(T)$.

Remark B.14. The restriction of a differential form on $\overline{\mathfrak{M}}(T)$ which is locally of the form (B.2) to $\mathfrak{M}(T)$ is holomorphic.

Thus, letting $j\colon \mathfrak{M}(T) \hookrightarrow \overline{\mathfrak{M}}(T)$ be the inclusion, we obtain a restriction morphism

$$\Omega^{\bullet}_{\overline{\mathfrak{M}}(T)}(\log \partial\overline{\mathfrak{M}}(T)) \to j_*\Omega^{\bullet}_{\mathfrak{M}(T)}.$$

Fact B.15 ([PS08, Proposition 4.3])**.** *The restriction morphism of Remark B.14 is a quasi-isomorphism and thus induces an isomorphism*

$$\mathbb{H}^k(\overline{\mathfrak{M}}(T); \Omega^{\bullet}_{\overline{\mathfrak{M}}(T)}(\log \partial\overline{\mathfrak{M}}(T))) \cong \mathbb{H}^k(\overline{\mathfrak{M}}(T); j_*\Omega_{\mathfrak{M}(T)}) \cong H^k(\mathfrak{M}(T))$$

for all $k \in \mathbb{N}$*.*

Now we fix $S \in \mathcal{T}_n$, $S \succcurlyeq T$ and want to define the residue of a logarithmic form on $(\overline{\mathfrak{M}}(T), \partial(\overline{\mathfrak{M}}(T))$ along S. For this, we first choose an enumeration $\alpha := (e_1, \ldots, e_m)$ of $\mathrm{In}(S, T)$.

Construction B.16. Let $x \in \overline{\mathfrak{M}}(S)$. Let T_i denote the n-tree obtained from S by contracting all the edges e_j with $j \in \{1, \ldots, m\}\backslash\{i\}$. We say that a chart centered at x is *compatible with* α if the stratification of $\overline{\mathfrak{M}}(T)$ corresponds to the standard stratification of Construction B.11 on that chart and under this correspondence, $\overline{\mathfrak{M}}(T_i)$, $i \in \{1, \ldots m\}$, corresponds to the hyperplane $\{(z_1, \ldots, z_k) \in \mathbb{C}^k \mid z_i = 0\}$. Note that compatible charts always exist by Fact B.12 and under a compatible chart, $\overline{\mathfrak{M}}(S)$ corresponds to $H := \{(z_1, \ldots, z_k) \in \mathbb{C}^k \mid z_i = 0 \text{ for } i \in \{1, \ldots, m\}\}$.

Using the local description of Definition B.13, we can write the restriction ω of a homogeneous form in $\Omega^p_{\overline{\mathfrak{M}}(T)}(\log \partial\overline{\mathfrak{M}}(T))$ to a compatible chart as

$$\omega = \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_m}{z_m} \wedge \eta_\omega + \eta'_\omega \tag{B.3}$$

where η_ω is not divisible by any of $\frac{dz_1}{z_1}, \ldots, \frac{dz_m}{z_m}$ and η'_ω is not divisible by $\frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_m}{z_m}$. Now we set $\operatorname{Res}^\alpha_{S,T}(\omega) = \eta_\omega|_H$.

This assignment is compatible with chart transitions as long as the charts are compatible with α. Further note that $\operatorname{Res}^\alpha_{S,T}(\omega)$ is a logarithmic form on $(\overline{\mathfrak{M}}(S), \partial\overline{\mathfrak{M}}(S))$. Thus, letting $j_{S,T} \colon \overline{\mathfrak{M}}(S) \hookrightarrow \overline{\mathfrak{M}}(T)$ be the inclusion, we obtain a morphism

$$\operatorname{Res}^\alpha_{S,T} \colon \Omega^p_{\overline{\mathfrak{M}}(T)}(\log \partial\overline{\mathfrak{M}}(T)) \to (j_{S,T})_* \Omega^{p-m}_{\overline{\mathfrak{M}}(S)}(\log \partial\overline{\mathfrak{M}}(S))$$

which we call *residue morphism with respect to α.*

Remark B.17. The Leibniz rule yields

$$\operatorname{Res}^\alpha_{S,T} \circ d_{\Omega^\bullet_{\overline{\mathfrak{M}}(T)}(\log \partial\overline{\mathfrak{M}}(T))} = (-1)^m \cdot d_{(j_{S,T})_*\Omega^\bullet_{\overline{\mathfrak{M}}(S)}(\log \partial\overline{\mathfrak{M}}(S))} \circ \operatorname{Res}^\alpha_{S,T}.$$

Thus, passing to (hyper)cohomology and using Fact B.15, we obtain residue maps

$$H^\bullet(\mathfrak{M}(T)) \to H^{\bullet - m}(\mathfrak{M}(S))$$

w. r. t. α which we also denote by $\operatorname{Res}^\alpha_{S,T}$.

$\operatorname{Res}^\alpha_{S,T}$ does indeed depend on the choice of an enumeration α of $\operatorname{In}(S,T)$:

Remark B.18. Let $\beta = (e_{i_1}, \ldots e_{i_m})$ be another enumeration of $\operatorname{In}(S,T)$ and let $\sigma_{\alpha,\beta} \in \Sigma_m$ denote the permutation mapping $j \in \{1, \ldots, m\}$ to i_j.

Then, for a local section ω as in (B.3) we have

$$\begin{aligned}\omega &= \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_m}{z_m} \wedge \eta_\omega + \eta'_\omega \\ &= \frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_m}}{z_{i_m}} \wedge (\operatorname{sgn}(\sigma_{\alpha,\beta}) \cdot \eta_\omega) + \eta''_\omega\end{aligned}$$

where η''_ω is not divisible by $\frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_m}}{z_{i_m}}$. As the first term is used to compute $\operatorname{Res}^\alpha_{S,T}$ and the second term is used to compute $\operatorname{Res}^\beta_{S,T}$, we obtain

$$\operatorname{Res}^\beta_{S,T} = \operatorname{sgn}(\sigma_{\alpha,\beta}) \cdot \operatorname{Res}^\alpha_{S,T}.$$

We can, however, get rid of these sign ambiguities by "twisting" by a sign representation:

Remark B.19. Let $\beta = (e_{i_1}, \ldots e_{i_m})$ be another enumeration of $\operatorname{In}(S,T)$ and $\sigma_{\alpha,\beta}$ as in Remark B.18. Then, for $c \in H^\bullet(\mathfrak{M}(T))$ we have the equality

$$\begin{aligned}&(e_{i_1} \wedge \ldots \wedge e_{i_m}) \otimes \operatorname{Res}^\beta_{S,T}(c) \\ =&(\operatorname{sgn}(\sigma_{\alpha,\beta}) \cdot (e_1 \wedge \ldots \wedge e_m)) \otimes (\operatorname{sgn}(\sigma_{\alpha,\beta}) \cdot \operatorname{Res}^\alpha_{S,T}(c)) \\ =&(e_1 \wedge \ldots \wedge e_m) \otimes \operatorname{Res}^\alpha_{S,T}(c)\end{aligned}$$

in $\det(\mathbb{C}^{\operatorname{In}(S,T)}) \otimes H^{\bullet - m}(\mathfrak{M}(S))$.

Definition B.20. Define the *(absolute) residue morphism* via

$$\begin{aligned}\mathrm{Res}_{S,T}\colon H^\bullet(\mathfrak{M}(T)) &\to \det(\mathbb{C}^{\mathrm{In}(S,T)}) \otimes H^{\bullet-|\mathrm{In}(S,T)|}(\mathfrak{M}(S))\\ c &\mapsto (e_1 \wedge \ldots \wedge e_m) \otimes \mathrm{Res}^{\alpha}_{S,T}(c).\end{aligned}$$

By Remark B.19, this definition does not depend on the choice of the enumeration α.

B.3 Properties of Residue Morphisms

Now we observe that absolute residue maps satisfy some compatibility relations with other structures mentioned above. In Subsection 2.1, these will be crucial in proving the cooperad axioms for the gravity cooperad.

Proposition B.21 (*compatibility with the group action*)**.** *Assume that T has a single internal edge e, i. e. $T \geqslant_1 C_n$. Let $\sigma \in \Sigma_n$. Then the action of σ on $\overline{\mathfrak{M}}_n$ takes the stratum $\mathfrak{M}(T)$ to the stratum $\mathfrak{M}(\sigma(T))$, where $\sigma(T)$ is the image of T under the action of σ on $\mathcal{T}_n$ given by permuting the labels (cf. Remark A.6). In particular, we have an isomorphism $\sigma_*\colon \mathfrak{M}(T) \cong \mathfrak{M}(\sigma(T))$. Moreover, since the underlying trees of T and $\sigma(T)$ coincide, we can identify* $\mathrm{In}(\sigma(T))$ *with* $\{e\} = \mathrm{In}(T)$.

With these notations, we have a commutative diagram

$$\begin{tikzcd} H^\bullet(\mathfrak{M}_n) \arrow[rr, "\sigma^*"] \arrow[d, "\mathrm{Res}_{\sigma(T),C_n}"'] & & H^\bullet(\mathfrak{M}_n) \arrow[d, "\mathrm{Res}_{T,C_n}"] \\ \det(\mathbb{C}^{\{e\}}) \otimes H^{\bullet-1}(\mathfrak{M}(\sigma(T))) \arrow[rr, "\mathrm{id}\otimes\sigma^*"'] & & \det(\mathbb{C}^{\{e\}}) \otimes H^{\bullet-1}(\mathfrak{M}(T)) \end{tikzcd}.$$

Proof. First we note that since $\sigma\colon \overline{\mathfrak{M}}_n \to \overline{\mathfrak{M}}_n$ maps strata to strata, pulling back along σ yields a morphism

$$\sigma^*\colon (j_{\sigma(S),C_n})_* \Omega^\bullet_{\overline{\mathfrak{M}}(\sigma(S))}(\log \partial\overline{\mathfrak{M}}(\sigma(S))) \to (j_{S,C_n})_* \Omega^\bullet_{\overline{\mathfrak{M}}(S)}(\log \partial\overline{\mathfrak{M}}(S))$$

for all $S \geqslant \mathcal{T}_n$ which induces $\sigma*\colon H^\bullet(\mathfrak{M}(\sigma(S))) \to H^\bullet(\mathfrak{M}(S))$ on cohomology.

Now we consider (e) both as an enumeration of $\mathrm{In}(T)$ and as an enumeration of $\mathrm{In}(\sigma(T))$. After trivializing $\det(\mathbb{C}^{\{e\}})$ via (e), we only need to show that on the level of logarithmic forms, the diagram

$$\begin{tikzcd} \Omega^\bullet_{\overline{\mathfrak{M}}_n}(\log \partial\overline{\mathfrak{M}}_n) \arrow[rr, "\sigma^*"] \arrow[d, "\mathrm{Res}^{(e)}_{\sigma(T),C_n}"'] & & \Omega^\bullet_{\overline{\mathfrak{M}}_n}(\log \partial\overline{\mathfrak{M}}_n) \arrow[d, "\mathrm{Res}^{(e)}_{T,C_n}"] \\ (j_{\sigma(T),C_n})_* \Omega^{\bullet-1}_{\overline{\mathfrak{M}}(\sigma(T))}(\log \partial\overline{\mathfrak{M}}(\sigma(T))) \arrow[rr, "\sigma^*"'] & & (j_{T,C_n})_* \Omega^{\bullet-1}_{\overline{\mathfrak{M}}(T)}(\log \partial\overline{\mathfrak{M}}(T)) \end{tikzcd}$$

commutes.

Note that for $x \in \overline{\mathfrak{M}}(T)$, if γ is a chart centered at x compatible with (e) as an enumeration of $\mathrm{In}(T)$, then $\gamma \circ \sigma^{-1}$ is a chart centered at $\sigma(x) \in \overline{\mathfrak{M}}(\sigma(T))$ compatible with (e) as an enumeration of $\mathrm{In}(\sigma(T))$. Thus, since residues on the both sides of the above diagram are locally given by "omitting the first coordinate" in a compatible chart, the two compositions coincide. $\square$

Proposition B.22 (*compatibility with product decompositions*)**.** *Let $T' \in \mathcal{T}_n$, $m \in \mathbb{N}_+$, $R, R' \in \mathcal{T}_m$ such that $T' \succcurlyeq T$ and $R' \succcurlyeq R$. Further let $i \in \{1, \ldots, n-1\}$. In this situation we have $T' \circ_i R' \succcurlyeq T \circ_i R$ and an identification $\mathrm{In}(T' \circ_i R', T \circ_i R) \cong \mathrm{In}(T', T) \dot\cup \mathrm{In}(R', R)$ which induces an isomorphism*

$$\phi_{T',T;R',R} \colon \det(\mathbb{C}^{\mathrm{In}(T \circ_i R', T \circ_i R)}) \to \det(\mathbb{C}^{\mathrm{In}(T',T')}) \otimes \det(\mathbb{C}^{\mathrm{In}(R',R)}).$$

With these notations, the diagram

$$\begin{tikzcd}
H^\bullet(\mathfrak{M}(T' \circ_i R)) \arrow[d, "\Psi^*_{T,R,i}"'] \arrow[r, bend left, "\mathrm{Res}_{T \circ_i R', T \circ_i R}"] & \det(\mathbb{C}^{\mathrm{In}(T' \circ_i R', T \circ_i R)}) \otimes H^{\bullet - |\mathrm{In}(T' \circ_i R', T \circ_i R)|}(\mathfrak{M}(T' \circ_i R')) \arrow[d, "\mathrm{id} \otimes \Psi^*_{T',R',i}"] \\
H^\bullet(\mathfrak{M}(T) \times \mathfrak{M}(R)) \arrow[dd, "\kappa"'] & \det(\mathbb{C}^{\mathrm{In}(T' \circ_i R', T \circ_i R)}) \otimes H^{\bullet - |\mathrm{In}(T' \circ_i R', T \circ_i R)|}(\mathfrak{M}(T') \times \mathfrak{M}(R')) \arrow[d, "\phi_{T',T;R',R} \otimes \kappa"] \\
 & \det(\mathbb{C}^{\mathrm{In}(T',T)}) \otimes \det(\mathbb{C}^{\mathrm{In}(R',R)}) \otimes H^{\bullet - |\mathrm{In}(T',T)|}(\mathfrak{M}(T')) \otimes H^{\bullet - |\mathrm{In}(R',R)|}(\mathfrak{M}(R')) \arrow[d, "\mathrm{id} \otimes \tau \otimes \mathrm{id}"] \\
H^\bullet(\mathfrak{M}(T)) \otimes H^\bullet(\mathfrak{M}(R)) \arrow[r, bend right, "\mathrm{Res}_{T',T} \otimes \mathrm{Res}_{R',R}"'] & \det(\mathbb{C}^{\mathrm{In}(T',T)}) \otimes H^{\bullet - |\mathrm{In}(T',T)|}(\mathfrak{M}(T')) \otimes \det(\mathbb{C}^{\mathrm{In}(R',R)}) \otimes H^{\bullet - |\mathrm{In}(R',R)|}(\mathfrak{M}(R'))
\end{tikzcd}$$

commutes.

Proof. Let $\alpha_1 = (e_1, \ldots, e_k)$ be an enumeration of $\mathrm{In}(T', T)$ and $\alpha_2 = (f_1, \ldots, f_l)$ an enumeration of $\mathrm{In}(R', R)$. Then we have an enumeration $(e_1, \ldots, e_k, f_1, \ldots, f_l)$ of $\mathrm{In}(T', T) \dot\cup \mathrm{In}(R', R)$ which induces an enumeration $\alpha = (g_1, \ldots, g_{k+l})$ of $\mathrm{In}(T' \circ_i R', T \circ_i R)$ such that

$$\phi_{T',T;R',R}(g_1 \wedge \ldots \wedge g_{k+l}) = (e_1 \wedge \ldots \wedge e_k) \otimes (f_1 \wedge \ldots \wedge f_l).$$

Now we want to analyze the situation on the level of logarithmic forms. For $\overline{\mathfrak{M}}(T) \times \overline{\mathfrak{M}}(R)$ resp. $\overline{\mathfrak{M}}(T') \times \overline{\mathfrak{M}}(R')$ we consider logarithmic forms $\Omega^\bullet_{\overline{\mathfrak{M}}(T) \times \overline{\mathfrak{M}}(R)}(\log \partial(\overline{\mathfrak{M}}(T) \times \overline{\mathfrak{M}}(R)))$ resp. $\Omega^\bullet_{\overline{\mathfrak{M}}(T') \times \overline{\mathfrak{M}}(R')}(\log \partial(\overline{\mathfrak{M}}(T') \times \overline{\mathfrak{M}}(R')))$ whose definition is analogous to the logarithmic forms of Definition B.13 and uses the induced stratification on the product spaces (cf. [PS08, Chapter 4]).

Moreover, let $(x_1, x_2) \in \overline{\mathfrak{M}}(T) \times \overline{\mathfrak{M}}(R)$ and γ_1 resp. γ_2 a chart of $\overline{\mathfrak{M}}(T)$ resp. $\overline{\mathfrak{M}}(R)$ centered at x_1 resp. x_2 that is compatible with α_1 resp. α_2.

Next, let ω_1 resp. ω_2 be a local section of $\mathrm{pr}_1^* \Omega^u_{\overline{\mathfrak{M}}(T)}(\log \partial \overline{\mathfrak{M}}(T))$ resp. $\mathrm{pr}_2^* \Omega^v_{\overline{\mathfrak{M}}(R)}(\log \partial \overline{\mathfrak{M}}(R))$ which is given by $\frac{dz_1}{z_1} \wedge \ldots \wedge \frac{dz_k}{z_k} \wedge \eta$ resp. $\frac{dw_1}{w_1} \wedge \ldots \wedge \frac{dw_l}{w_l} \wedge \mu$ in the coordinates of γ_1 resp. γ_2. Then, using the trivializations of determinants induced by the above mentioned enumerations and taking the sign introduced by the braiding τ into account, it is enough to show that for $\omega_1 \otimes \omega_2$ the compositions in the diagram

$$\begin{tikzcd}
(\Psi^{-1}_{T,R,i})^* \Omega^\bullet_{\overline{\mathfrak{M}}(T \circ_i R)}(\log \partial \overline{\mathfrak{M}}(T \circ_i R)) \arrow[r, bend left, "(\Psi^{-1}_{T,R,i})^*(\mathrm{Res}^{\alpha}_{T \circ_i R', T \circ_i R})"] & (\Psi^{-1}_{T,R,i})^* \left((j_{T' \circ_i R', T \circ_i R})_* \Omega^{\bullet - |\mathrm{In}(T' \circ_i R', T \circ_i R)|}_{\overline{\mathfrak{M}}(T' \circ_i R')}(\log \partial \overline{\mathfrak{M}}(T' \circ_i R')) \right) \\
\Omega^\bullet_{\overline{\mathfrak{M}}(T) \times \overline{\mathfrak{M}}(R)}(\log \partial (\overline{\mathfrak{M}}(T) \times \overline{\mathfrak{M}}(R))) \arrow[u, "(\Psi^{-1}_{T,R,i})^*"] & (j_{T',T} \times j_{R',R})_* \Omega^{\bullet - |\mathrm{In}(T' \circ_i R', T \circ_i R)|}_{\overline{\mathfrak{M}}(T') \times \overline{\mathfrak{M}}(R')}(\log \partial (\overline{\mathfrak{M}}(T') \times \overline{\mathfrak{M}}(R'))) \arrow[u, "(\Psi^{-1}_{T',R',i})^*"'] \\
\mathrm{pr}_1^* \Omega^\bullet_{\overline{\mathfrak{M}}(T)}(\log \partial \overline{\mathfrak{M}}(T)) \otimes \mathrm{pr}_2^* \Omega^\bullet_{\overline{\mathfrak{M}}(R)}(\log \partial \overline{\mathfrak{M}}(R)) \arrow[u, "\tilde{\kappa}"] \arrow[r, bend right, "\mathrm{pr}_1^*(\mathrm{Res}^{\alpha_1}_{T',T}) \otimes \mathrm{pr}_2^*(\mathrm{Res}^{\alpha_2}_{R',R})"'] & \begin{array}{c} \mathrm{pr}_1^* \left((j_{T',T})_* \Omega^{\bullet - |\mathrm{In}(T',T)|}_{\overline{\mathfrak{M}}(T')}(\log \partial \overline{\mathfrak{M}}(T')) \right) \otimes \\ \mathrm{pr}_2^* \left((j_{R',R})_* \Omega^{\bullet - |\mathrm{In}(R',R)|}_{\overline{\mathfrak{M}}(R')}(\log \partial \overline{\mathfrak{M}}(R')) \right) \end{array} \arrow[u, "\tilde{\kappa}"']
\end{tikzcd}$$

coincide up to the sign $(-1)^{(u-k) \cdot l}$ where $\tilde{\kappa}$ denotes the chain-level Künneth morphism given on the level of differential forms by $\zeta \otimes \xi \mapsto \zeta \wedge \xi$.

Note that the bottom-right composition maps $\omega_1 \otimes \omega_2$ to $(\Psi^{-1}_{T,R,i})^*(\eta \wedge \mu)$. Now, in order to make the chart $(\gamma_1 \times \gamma_2) \circ \Psi^{-1}_{T,R,i}$ compatible with α, one has to permute the last $u - k$ coordinates of γ_1 past the first l coordinates of γ_2, which introduces the sign $(-1)^{(u-k) \cdot l}$. Then $(\Psi^{-1}_{T,R,i})^*(\mathrm{Res}^{\alpha}_{T \circ_i R', T \circ_i R})$ "removes the coordinates $z_1, \ldots, z_k, w_1, \ldots, w_l$", so that the left-top composition maps $\omega_1 \otimes \omega_2$ to $(-1)^{(u-k) \cdot l} (\Psi^{-1}_{T,R,i})^*(\eta \wedge \mu)$ which has the required sign. $\square$

Proposition B.23 (*compatibility with sequences of edge contractions*)**.** *Let $R \geqslant S \geqslant T$ be sequence of internal edge contractions. Then the decomposition* $\mathrm{In}(R,T) = \mathrm{In}(S,T) \dot{\cup} \mathrm{In}(R,S)$ *yields an isomorphism*

$$\phi_{R,S,T} \colon \det(\mathbb{C}^{\mathrm{In}(S,T)}) \otimes \det(\mathbb{C}^{\mathrm{In}(R,S)}) \xrightarrow{\cong} \det(\mathbb{C}^{\mathrm{In}(R,T)})$$

such that the composition

$$
\begin{aligned}
& && H^\bullet(\mathfrak{M}(T)) \\
& \xrightarrow{\mathrm{Res}_{S,T}} && \det(\mathbb{C}^{\mathrm{In}(S,T)}) \otimes H^{\bullet-|\mathrm{In}(S,T)|}(\mathfrak{M}(S)) \\
& \xrightarrow{\mathrm{id}\otimes\mathrm{Res}_{R,S}} && \det(\mathbb{C}^{\mathrm{In}(S,T)}) \otimes \det(\mathbb{C}^{\mathrm{In}(R,S)}) \otimes H^{\bullet-|\mathrm{In}(S,T)|-|\mathrm{In}(R,S)|}(\mathfrak{M}(R)) \\
& \xrightarrow{\phi_{R,S,T}\otimes\mathrm{id}} && \det(\mathbb{C}^{\mathrm{In}(R,T)}) \otimes H^{\bullet-|\mathrm{In}(R,T)|}(\mathfrak{M}(R))
\end{aligned}
$$

is equal to $\mathrm{Res}_{R,T}$*.*

Proof. Let $\alpha_1 = (e_1, \ldots, e_k)$ be an enumeration of $\mathrm{In}(S,T)$ and $\alpha_2 = (f_1, \ldots, f_l)$ an enumeration of $\mathrm{In}(R,S)$, so that $\alpha = (e_1, \ldots, e_k, f_1, \ldots, f_l)$ is an enumeration of $\mathrm{In}(R,T)$.

Then $\mathrm{Res}^{\alpha_2}_{R,S} \circ \mathrm{Res}^{\alpha_1}_{S,T} = \mathrm{Res}^{\alpha}_{R,T}$ since on the level of logarithmic forms, both sides are locally given by "omitting first $k+l$ coordinates" on charts compatible with α.

Moreover, since

$$\phi_{R,S,T}((e_1 \wedge \ldots \wedge e_k) \otimes (f_1 \wedge \ldots \wedge f_l)) = e_1 \wedge \ldots \wedge e_k \wedge f_1 \wedge \ldots \wedge f_l,$$

this identity is compatible with the signs and thus yields the desired identity for the absolute residue morphisms. □

B.4 The Residue Spectral Sequence

We are now going to present a spectral sequence which relates the cohomology of $\mathfrak{M}_n$ with the cohomology of $\overline{\mathfrak{M}}_n$. Similar spectral sequences were used in [GK94] and [Get95], and more recently in [AP15] and [DV15] to establish Koszul duality relations for various versions of the gravity operad. The variant we use is from [DV15].

Fact B.24 ([DV15, Proposition 3.6])**.** *Consider the double complex of sheaves on* $\overline{\mathfrak{M}}_n$ *given by*

$$\mathscr{K}^{p,q} := \bigoplus_{\substack{T\in\mathcal{T}_n \\ |\mathrm{In}(T)|=p}} (j_T)_*\Omega^{q-p}_{\overline{\mathfrak{M}}(T)}(\log \partial\overline{\mathfrak{M}}(T))$$

where the vertical differential is given by the usual de Rham differential and the horizontal differential is given by the sum of the residue morphisms

$$(j_T)_*\left(\mathrm{Res}^{(e)}_{T',T}\right) : (j_T)_*\Omega^{q-p}_{\overline{\mathfrak{M}}(T)}(\log \partial\overline{\mathfrak{M}}(T)) \to (j_{T'})_*\Omega^{q-(p+1)}_{\overline{\mathfrak{M}}(T')}(\log \partial\overline{\mathfrak{M}}(T'))$$

for all $T' \geqslant_1 T$ *such that* $T \cong T'/e$*.*

This double complex induces a spectral sequence

$$E_1^{p,q} = \bigoplus_{\substack{T\in\mathcal{T}_n \\ |\mathrm{In}(T)|=p}} H^{q-p}(\mathfrak{M}(T)) \Rightarrow H^{p+q}(\overline{\mathfrak{M}}_n)$$

of representations of Σ_n *computing the cohomology of* $\overline{\mathfrak{M}}_n$*.*

In fact, using additional information about Hodge structures, one can show that this spectral sequence collapses on the E_2-page and describe its E_∞-page.

Fact B.25 ([DV15, Remark 3.12], [Get95, Section 3]). *The residue spectral sequence of Fact B.24 collapses on the second page and the E_∞-page is concentrated on the line $p = q$.*

In particular, $H^\bullet(\overline{\mathfrak{M}}_n)$ is concentrated in even degrees and for $q \in \mathbb{N}$, the transition from the E_1-page to the $E_2 = E_\infty$-page yields an exact sequence

$$0 \to H^q(\mathfrak{M}_n) \xrightarrow{d_1^{0,q}} \bigoplus_{\substack{T \in \mathcal{T}_n \\ |\operatorname{In}(T)|=1}} H^{q-1}(\mathfrak{M}(T)) \xrightarrow{d_1^{1,q}} \dots \xrightarrow{d_1^{q-1,q}} \bigoplus_{\substack{T \in \mathcal{T}_n \\ |\operatorname{In}(T)|=q}} H^0(\mathfrak{M}(T)) \to H^{2q}(\overline{\mathfrak{M}}_n) \to 0.$$

This exact sequence is used in Subsection 2.2 to establish the Koszul duality between the gravity operad and the hypercommutative operad.

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