Simplicial homology of the real projective plane and the Klein bottle

Aras Ergus

June 27, 2019

This document contains two examples of simplicial homology computations. They were originally written as model solutions for some exercises in a course on algebraic topology given by Kathryn Hess and showcase some techniques (such as the Mayer–Vietoris sequence and excision) that were introduced in the class around the time those exercises appeared.

Contents

1 The real projective plane

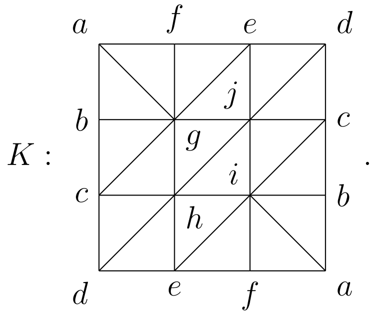
In this section, we will find a(n abstract) simplicial complex K whose realization is homeomorphic to the real projective plane $\mathbb{R}P^2$ and compute its simplicial homology using a Mayer–Vietoris sequence.

In the following, we freely use some usual conventions and abuses of notation such as representing abstract simplicial complexes and their labelings with pictures, using the alphabetical ordering of the vertices for homology calculations, writing v instead of $\{v\}$ for a 0-simplex etc.

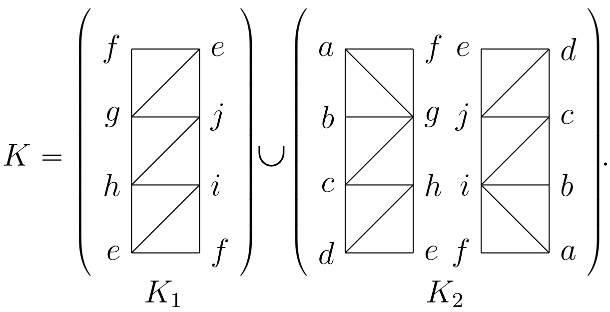
Fixing possible sign issues is left as an exercise to the reader.

The complex and the decomposition

Here is a possible way to realize $\mathbb{R}P^2$ as a simplicial complex:[1]



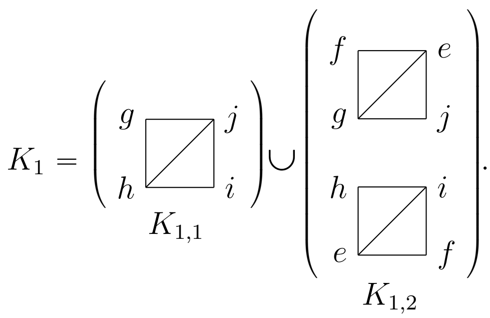
To use the Mayer–Vietoris sequence, we want to write K as the union of two subcomplexes whose homology (and that of their intersection) we know well (or can compute easily). One of the many possibilities to do this is as follows:



First subcomplex

Note that K_1 is a representation of the Möbius strip. You may have seen that its homology is isomorphic to that of $\partial\Delta^2$, but we will compute it to have another demonstration of how to use the Mayer–Vietoris sequence and because it will be important to know an explicit generator of $H_1(K_1) \cong H_1(\partial\Delta^2) \cong \mathbb{Z}$.

Here is the decomposition of K_1 that we will use to compute its homology:

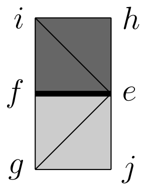


[1]There is actually a simplicial complex with only ten 2-simplicies that realizes $\mathbb{R}P^2$, but we'll stick to the one above because it's somewhat more straightforward to come up with: One can obtain $\mathbb{R}P^2$ from a square by identifying "antipodal" points of its boundary, and inspired by how one realizes the cylinder as a simplicial complex, one can subdivide that square into three "layers" vertically and horizontally to obtain the simplicial complex above.

Note that $K_{1,1}$ is the union of two copies of Δ^2 whose intersection is isomorphic to Δ^1. Thus it is acyclic as the union of two acyclic subcomplexes whose intersection is also acyclic.

Doing the identifications given by the labeling, we see that $K_{1,2}$ can be written as the union of two copies of $K_{1,1}$ whose intersection is isomorphic to Δ^1:

$K_{1,2}$:



.

Hence $K_{1,2}$ is also acyclic.

The intersection $K_{1,1} \cap K_{1,2}$ consists of two disjoint copies of Δ^1 (namely those spanned by $\{g, j\}$ and $\{h, i\}$). Thus $H_n(K_{1,1} \cap K_{1,2}) \cong 0$ for $n > 0$ and $H_0(K_{1,1} \cap K_{1,2}) \cong \mathbb{Z}^2$ is a free abelian group on the generators $[g]$ and $[h]$.

We can now compute the homology of K_1. First we note that K_1 is connected, so $H_0(K_1) \cong \mathbb{Z}$ (which can also be seen from the Mayer–Vietoris sequence).

To calculate $H_1(K_1)$, we have a look at the corresponding segment of the Mayer–Vietoris sequence:

$$0 \cong H_1(K_{1,1}) \oplus H_1(K_{1,2}) \to H_1(K_1) \xrightarrow{\partial_1} H_0(K_{1,1} \cap K_{1,2}) \xrightarrow{\phi_0} H_0(K_{1,1}) \oplus H_0(K_{1,2}).$$

This means that ∂_1 is injective and thus an isomorphism onto its image $\operatorname{im} \partial_1 = \ker \phi_0$.

To determine $\ker \phi_0$, we note that $[h] = [g]$ in $H_0(K_{1,1})$ and $H_0(K_{1,2})$, so

$$\phi_0(m[g] + n[h]) = ((m+n)[g], -(m+n)[g]) \in H_0(K_{1,1}) \oplus H_0(K_{1,2})$$

which is zero if and only if $m = -n$. Hence we have

$$\ker \phi_0 = \{-k[g] + k[h] \mid k \in \mathbb{Z}\} = \mathbb{Z} \cdot ([h] - [g]) \subseteq H_0(K_{1,1} \cap K_{1,2}) = \mathbb{Z} \cdot [g] \oplus \mathbb{Z} \cdot [h].$$

Thus $H_1(K_1) \cong \mathbb{Z}$ and the preimage of $[h] - [g]$ under ∂_1 is a generator.

Intuitively speaking, this preimage is represented by two sequences of edges connecting g and h in $K_{1,1}$ resp. $K_{1,2}$ such that their union is a cycle in K_1. An example of this would be taking $(\{g, h\})$ in $K_{1,1}$ and $(\{f, g\}, \{e, f\}, \{e, h\})$ in $K_{1,2}$, which would yield the generator $[\{f, g\} + \{g, h\} - \{e, h\} + \{e, f\}] \in H_1(K_1)$ after choosing appropriate signs.

In order to be more precise about this, we have to recall how ∂_1 is defined using the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_1(K_{1,1} \cap K_{1,2}) & \xrightarrow{\varphi_1} & C_1(K_{1,1}) \oplus C_1(K_{1,2}) & \xrightarrow{\varrho_1} & C_1(K_1) & \longrightarrow & 0 \\
 & & \big\downarrow d_1^{K_{1,1} \cap K_{1,2}} & & \big\downarrow d_1^{K_{1,1}} \oplus d_1^{K_{1,2}} & & \big\downarrow d_1^{K_1} & & \\
0 & \longrightarrow & C_0(K_{1,1} \cap K_{1,2}) & \xrightarrow[\varphi_0]{} & C_0(K_{1,1}) \oplus C_0(K_{1,2}) & \xrightarrow[\varrho_0]{} & C_0(K_1) & \longrightarrow & 0
\end{array}.$$

Namely, given a 1-cycle η in K_1, one lifts η along ϱ_1, checks that the image of the lift under $d_1^{K_{1,1}} \oplus d_1^{K_{1,2}}$ comes from a 0-cycle η' in $C_0(K_{1,1} \cap K_{1,2})$ and sets $\partial_1([\eta])$ to be $[\eta'] \in H_0(K_{1,1} \cap K_{1,2})$.

Hence, if we can find $\alpha \in C_1(K_{1,2})$ and $\beta \in C_1(K_{1,2})$ such that $(d_1^{K_{1,1}}(\alpha), d_1^{K_{1,2}}(\beta)) = (h-g, g-h) = \varphi_0(h-g)$ and $\gamma := \varrho_1(\alpha, \beta) = \alpha + \beta \in \ker d_1^{K_1}$, we will have $\partial_1([\gamma]) = [h] - [g]$, which means that $[\gamma]$ is a generator of $H_1(K_1)$.

To realize the example from above, we set $\alpha = \{g, h\}$ and $\beta = -\{e, h\} + \{e, f\} + \{f, g\}$. Then we indeed have $d_1^{K_{1,1}}(\alpha) = h-g$ and $d_1^{K_{1,2}}(\beta) = e-h+f-e+g-f = g-h$. Moreover, a straightforward calculation shows that $\gamma := \varrho_1(\alpha, \beta) = \{f, g\} + \{g, h\} - \{e, h\} + \{e, f\}$ is a cycle, so $[\gamma] = [\{f, g\} + \{g, h\} - \{e, h\} + \{e, f\}]$ is indeed a generator of $H_1(K_1)$.

Next, we see that $H_2(K_1)$ is "squeezed between trivial groups" in the MV sequence:

$$0 \cong H_2(K_{1,1}) \oplus H_2(K_{1,2}) \to H_2(K_1) \to H_1(K_{1,1} \cap K_{1,2}) \cong 0,$$

so it also is trivial. Moreover, $H_n(K_1) \cong 0$ for $n > 2$ as K_1 is a 2-dimensional complex.

All in all, we have calculated that

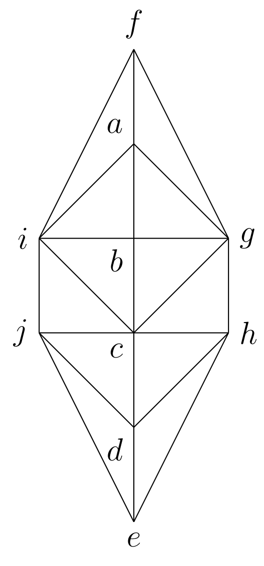
$$H_n(K_1) \cong \begin{cases} \mathbb{Z} & n \in \{0, 1\} \\ 0 & \text{otherwise} \end{cases}$$

where $H_1(K_1) = \mathbb{Z} \cdot [\{f, g\} + \{g, h\} - \{e, h\} + \{e, f\}]$.

Second subcomplex

After doing the identifications given by the labeling, K_2 looks as follows:

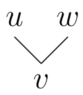
K_2 :



.

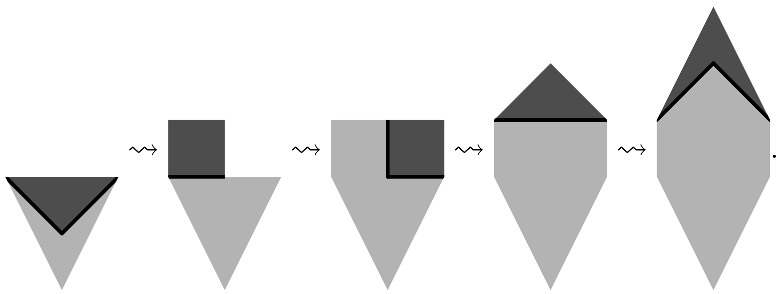
The picture makes it evident that $|K_2|$ is homeomorphic to a disk and we will show that K_2 is indeed acyclic by decomposing it into two acyclic subcomplexes whose intersection is acyclic.

First we note that the simplicial complex

$L :$ 

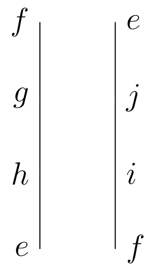
is acyclic because it is the union of two copies of Δ^1 whose intersection is isomorphic to Δ^0.

Now we can iteratively build K_2 by starting with a complex isomorphic to the acyclic complex $K_{1,1}$ from above and in each step adding a copy of $K_{1,1}$ in a way that the intersection is isomorphic to Δ^1 or L (thus also acyclic), which means that each complex in the sequence is acyclic:



The intersection and the final MV sequence

The intersection $K_1 \cap K_2$ is given by

$K_1 \cap K_2 :$ 

which represents a hexagon with vertices f, g, h, e, j, i after doing the identifications indicated by the labeling.

We refrain from computing its homology here which can be done directly or using a Mayer–Vietoris sequence. The result is

$$H_n(K_1 \cap K_2) \cong \begin{cases} \mathbb{Z} & n \in \{0, 1\} \\ 0 & \text{otherwise} \end{cases}$$

where $H_1(K_1 \cap K_2)$ is generated by the class of $\theta := \{f, g\} + \{g, h\} - \{e, h\} + \{e, j\} - \{i, j\} - \{f, i\}$, i. e. a generating cycle is given by "going around the circle once".

Now we start computing the homology of K. Since K is connected, $H_0(K) \cong \mathbb{Z}$.

To compute $H_1(K)$, we will analyze the following segment of the Mayer–Vietoris sequence:

$$H_1(K_1 \cap K_2) \xrightarrow{\phi_1} H_1(K_1) \oplus H_1(K_2) \xrightarrow{\rho_1} H_1(K) \xrightarrow{\partial_1} H_0(K_1 \cap K_2) \xrightarrow{\phi_0} H_0(K_1) \oplus H_0(K_2).$$

Using the homology class of g as a generator of 0-th homology groups of K_1, K_2 and $K_1 \cap K_2$, we see that

$$\begin{aligned} \mathbb{Z} \cong H_0(K_1 \cap K_2) &\xrightarrow{\phi_0} H_0(K_1) \oplus H_0(K_2) \cong \mathbb{Z}^2 \\ m[g] &\mapsto (m[g], -m[g]) \end{aligned}$$

is injective, i. e. $\ker \phi_0 = 0$.

Hence, by exactness, $\operatorname{im} \partial_1 = \ker \phi_0 = 0$. This yields, again by exactness, $H_1(K) = \ker \partial_1 = \operatorname{im} \rho_1$. Using that $\ker \rho_1 = \operatorname{im} \phi_1$, this means that $H_1(K) \cong \operatorname{coker} \phi_1$ by the first isomorphism theorem.

Now ϕ_1 is a homomorphism

$$\mathbb{Z} \cong H_1(K_1 \cap K_2) \to H_1(K_1) \oplus H_1(K_2) \cong H_1(K_1) \oplus 0 \cong \mathbb{Z},$$

so it maps the generator $[\theta] = [\{f, g\} + \{g, h\} - \{e, h\} + \{e, j\} - \{i, j\} - \{f, i\}]$ of $H_1(K_1 \cap K_2)$ to a multiple $k \cdot [\gamma]$ of the generator $[\gamma] = [\{f, g\} + \{g, h\} - \{e, h\} + \{e, f\}]$ of $H_1(K_1)$ and thus its image is $k \cdot \mathbb{Z} \cdot [\gamma]$, which means that its cokernel is isomorphic to $\mathbb{Z}/k\mathbb{Z}$.

Intuitively speaking, one can say that "the cycle $\{f, g\} + \{g, h\} - \{e, h\} + \{e, j\} - \{i, j\} - \{f, i\}$ goes around the Möbius strip K_1 twice", so k must be 2. This can be made precise as follows:

Let $\gamma' := \{e, j\} - \{i, j\} - \{f, i\} - \{e, f\} \in C_1(K_1 \cap K_2) \subseteq C_1(K_1)$. Note that γ' is a cycle in K_1 and that $\theta = \gamma + \gamma'$, so $[\theta] = [\gamma] + [\gamma']$ in $H_1(K_1)$. Therefore it is enough to show that $[\gamma] = [\gamma']$, i. e. $[\gamma - \gamma'] = 0$, in $H_1(K_1)$. Also this has a geometric interpretation: In the representation of K_1 as a rectangle whose top and bottom edge are appropriately identified, $\gamma - \gamma'$ corresponds to the boundary of the rectangle, so it is the image of an appropriate sum of the 2-simplices in the rectangle under $d_2^{K_1}$:

$$\begin{aligned} &= d_2^{K_1}(\{e, f, g\} + \{e, g, j\} + \{g, h, j\} + \{h, i, j\} - \{e, h, i\} + \{e, f, i\}) \\ &= (\{f, g\} - \{e, g\} + \{e, f\}) + (\{g, j\} - \{e, j\} + \{e, g\}) + (\{h, j\} - \{g, j\} + \{g, h\}) + \\ &\qquad (\{i, j\} - \{h, j\} + \{h, i\}) - (\{h, i\} - \{e, i\} + \{e, h\}) + (\{f, i\} - \{e, i\} + \{e, f\}) \\ &= (\{f, g\} + \{e, f\}) + (-\{e, j\}) + (\{g, h\}) + (\{i, j\}) - (\{e, h\}) + (\{f, i\} + \{e, f\}) \\ &= (\{f, g\} + \{g, h\} - \{e, h\} + \{e, f\}) + (-\{e, j\} + \{i, j\} + \{f, i\} + \{e, f\}) \\ &= \gamma - \gamma'. \end{aligned}$$

Hence $\phi_1([\theta])$ indeed corresponds to $[\gamma'] + [\gamma] = [\gamma] + [\gamma] = 2[\gamma]$, so $H_1(K) \cong \mathbb{Z}/2\mathbb{Z}$.

Note that the calculation above also shows that $\ker \phi_1 = 0$ as ϕ_1 is essentially given by multiplication by 2. Hence, looking at the exact sequence

$$0 \cong H_2(K_1) \oplus H_1(K_2) \xrightarrow{\rho_2} H_2(K) \xrightarrow{\partial_2} H_1(K_1 \cap K_2) \xrightarrow{\phi_1} H_1(K_1) \oplus H_1(K_2),$$

we see that $0 = \ker \phi_1 = \operatorname{im} \partial_2$, so $H_2(K) = \ker \partial_2 = \operatorname{im} \rho_2 = 0$.

Moreover, as K is a 2-dimensional simplicial complex, we have $H_n(K) \cong 0$ for all $n > 2$.

All in all, we obtain

$$H_n(K) \cong \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/2\mathbb{Z} & n = 1 \\ 0 & \text{otherwise} \end{cases} .$$

2 The Klein bottle

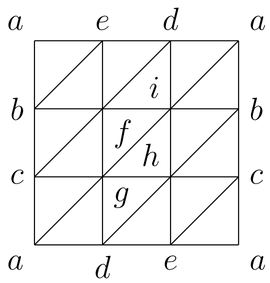
In this second section, we will describe another simplicial complex K whose realization is homeomorphic to the Klein bottle, and compute its homology using the (simplicial) excision theorem and the long exact sequence for a couple.

This section is somewhat more sketchy than the first one. As before, there may still be some sign or labeling mistakes left in the text.

The complex and the decomposition

Here is a possible way to realize the Klein bottle as a simplicial complex:

K :

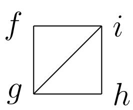


.

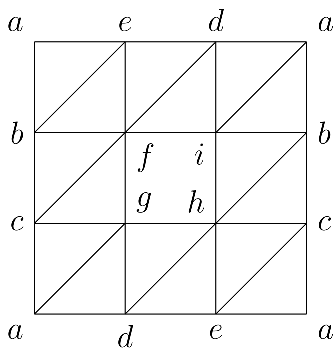
The idea of our computation is finding a decomposition $K = K_1 \cup K_2$ into subcomplexes such that we can understand $H_\bullet(K_2)$ and $H_\bullet(K, K_2)$ (where the latter can be identified with $H_\bullet(K_1, K_1 \cap K_2)$ via excision) well and deduce what $H_\bullet(K)$ must be by analyzing the long exact sequence for the couple (K, K_2).

To this end, we let

$K_1 :=$

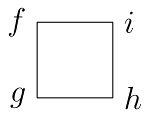


and $K_2 :=$



.

Note that we have

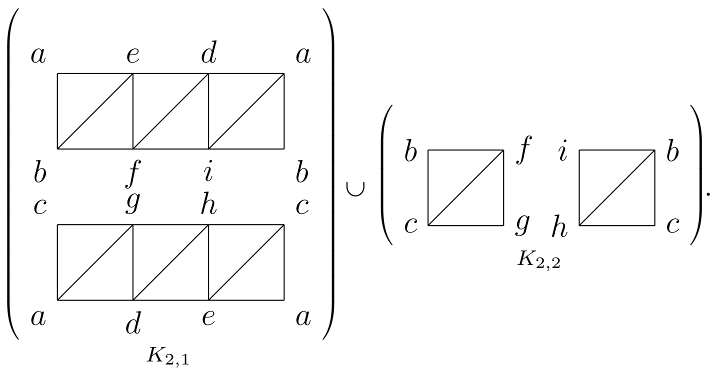
$$K_1 \cap K_2 =$$ 

.

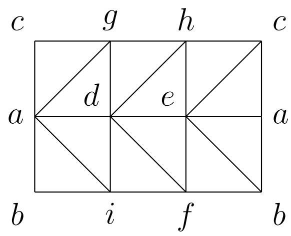
Homology of the subcomplex

Intuitively speaking, "K_2 is just a thickened version of two loops $\{a, b\} \cup \{a, c\} \cup \{a, c\}$ and $\{a, e\} \cup \{d, e\} \cup \{a, d\}$ joined at a", so we can expect to have $H_1(K_2) \cong \mathbb{Z}$, $H_1(K_2) \cong \mathbb{Z}^2$ and $H_n(K_2) \cong 0$ for $n \notin \{0, 1\}$.

To be more precise, we can use a Mayer–Vietoris sequence to compute $H_\bullet(K_2)$. We start with a decomposition

$$K_2 =$$ 

Note that

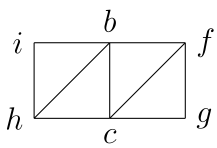
$$K_{2,1} =$$ 

realizes to a cylinder. We will just use that

$$H_n(K_{2,1}) \cong \begin{cases} \mathbb{Z} & n \in \{0, 1\} \\ 0 & \text{otherwise} \end{cases}$$

and that $[c]$ resp. $\gamma_1 := [\{a, d\} + \{d, e\} - \{a, e\}]$ is a generator of $H_0(K_{2,1})$ resp. $H_1(K_{2,1})$ without calculating these groups explicitly.

Moreover,

$$K_{2,2} =$$ 

is acyclic (as it can be built from acyclic complexes with acyclic intersection at each step). Hence

$$H_n(K_{2,1}) \cong \begin{cases} \mathbb{Z} \cdot [c] \cong \mathbb{Z} & n = 0 \\ 0 & \text{otherwise} \end{cases}.$$

The intersection

$$K_{2,1} \cap K_{2,2} = \begin{array}{c} i \overset{b}{\text{———————}} f \\ \\ h \underset{c}{\text{———————}} g \end{array}$$

is a disjoint union of two acyclic complexes, so we have

$$H_n(K_{2,1}) \cong \begin{cases} \mathbb{Z} \cdot [b] \oplus \mathbb{Z} \cdot [c] \cong \mathbb{Z}^2 & n = 0 \\ 0 & \text{otherwise} \end{cases}.$$

For $n > 1$, the Mayer–Vietoris sequence yields an exact sequence

$$0 \cong H_n(K_{2,1}) \oplus H_n(K_{2,2}) \to H_n(K_2) \to H_{n-1}(K_{2,1} \cap K_{2,1}) \cong 0,$$

so we indeed have $H_n(K_2) \cong 0$ for $n > 1$. Moreover, $H_0(K_2) = \mathbb{Z} \cdot [c] \cong \mathbb{Z}$ since K_2 is connected.

Hence, the Mayer–Vietoris sequence for $n \leqslant 1$ looks like:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathbb{Z} \cdot \gamma_1 & \longrightarrow & H_1(K_2) & \xrightarrow{\partial_1} & \mathbb{Z} \cdot [b] \oplus \mathbb{Z} \cdot [c] & \xrightarrow{\phi_1} & \mathbb{Z}[c_{K_{2,1}}] \oplus \mathbb{Z}[c_{K_{2,1}}] & \longrightarrow \mathbb{Z} \cdot [c] & \longrightarrow 0 \\ & & & & & & [b] & \longmapsto & ([c_{K_{2,1}}], -[c_{K_{2,2}}]) & & \\ & & & & & & [c] & \longmapsto & ([c_{K_{2,1}}], -[c_{K_{2,2}}]) & & \end{array}$$

where $c_{K_{2,k}}$ for $k \in \{1, 2\}$ denotes the vertex c regarded as a 0-simplex of $K_{2,k}$. This yields an exact sequence

$$0 \to \mathbb{Z} \cdot \gamma_1 \to H_1(K_2) \xrightarrow{\partial_1} \ker \phi_1 = \mathbb{Z} \cdot [c - d] \to 0.$$

Now, unwinding the definition of the boundary map, one can check that $\gamma_2 := [\{a, b\} + \{b, c\} - \{a, c\}] \in H_1(K_2)$ is a preimage of $[c-d] \in H_0(K_{2,1} \cap K_{2,2})$ under ∂_1.[2] Thus, sending $[c-b]$ to γ_2 yields a section of ∂_1 and hence an isomorphism $H_1(K_2) \cong \mathbb{Z} \cdot \gamma_1 \oplus \mathbb{Z} \cdot \gamma_2 \cong \mathbb{Z}^2$.[3]

[2] Intuitively, γ_1 is chosen to be the sum of two 1-chains which connect b and c in $K_{2,1}$ resp. $K_{2,2}$.

[3] In fact, every short exact sequence of abelian groups that is of the form $0 \to \mathbb{Z} \to A \to \mathbb{Z} \to 0$ splits for algebraic reasons and thus the middle term is always isomorphic to $\mathbb{Z}^2$.

Relative homology

By excision, we have $H_\bullet(K, K_2) \cong H_\bullet(K_1, K_1 \cap K_2)$. Now, $C_\bullet(K_1, K_1 \cap K_2)$ is rather simple:

$$\begin{array}{ccccccccc} \dots & & 3 & & 2 & & 1 & & 0 \\ \dots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}\cdot[\{f,g,i\}] \oplus \mathbb{Z}\cdot[\{g,h,i\}] & \xrightarrow{d_1} & \mathbb{Z}\cdot[\{g,i\}] & \longrightarrow & 0 \\ & & & & [\{f,g,i\}] & \longmapsto & [\{g,i\}] & & \\ & & & & [\{g,h,i\}] & \longmapsto & -[\{g,i\}] & & \end{array}$$

where $[-]$ denotes equivalence classes in $C_k(K_1, K_1 \cap K_2) = C_k(K_1)/C_k(K_1 \cap K_2)$.

Hence we have

$$H_n(K_1, K_1 \cap K_2) \cong \begin{cases} \operatorname{coker} d_1 \cong \mathbb{Z}\cdot[\{g,i\}]/\mathbb{Z}[\{g,i\}] \cong 0 & n = 1 \\ \ker d_1 \cong \mathbb{Z}\cdot[\{f,g,i\} + \{g,h,i\}] \cong \mathbb{Z} & n = 2 \\ 0 & \text{otherwise} \end{cases}.$$

Let $\alpha := [\{f,g,i\} + \{g,h,i\}]$ for future reference.

Long exact sequence of the couple

Note that $H_0(K) \cong \mathbb{Z}$ as K is connected. Moreover, we know that $H_n(K) \cong 0$ for $n > 2$ as K is a 2-dimensional complex. For $n \in \{1,2\}$ we consider the corresponding part of the long exact sequence of the pair (K, K_2):

$$\begin{array}{ccccccccccc} H_2(K_2) & \longrightarrow & H_2(K) & \longrightarrow & H_2(K,K_2) & \longrightarrow & H_1(K_2) & \longrightarrow & H_1(K) & \longrightarrow & H_1(K,K_2) \\ \wr\| & & \wr\| & & \wr\| & & \wr\| & & \wr\| & & \wr\| \\ H_2(K_2) & \longrightarrow & H_2(K) & \longrightarrow & H_2(K_1, K_1\cap K_2) & \longrightarrow & H_1(K_2) & \longrightarrow & H_1(K) & \longrightarrow & H_1(K_1, K_1 \cap K_2) \\ \wr\| & & \wr\| & & \wr\| & & \wr\| & & \wr\| & & \wr\| \\ 0 & \xrightarrow[\iota_2]{} & H_2(K) & \xrightarrow[\rho_2]{} & \mathbb{Z}\cdot\alpha & \xrightarrow[\partial_2]{} & \mathbb{Z}\cdot\gamma_1 \oplus \mathbb{Z}\cdot\gamma_2 & \longrightarrow & H_1(K) & \longrightarrow & 0 \end{array}$$

Unwinding definitions, one sees that $\partial_2(\alpha) = [\{f,g\} + \{g,h\} + \{h,i\} - \{f,i\}]$ which corresponds to "going around the inner square once". This cycle is equivalent to "going around the outer square" in K_2,[4] i. e.

$$\begin{aligned} \partial_2(\alpha) &= [\{a,b\} + \{b,c\} - \{a,c\} + \{a,d\} + \{d,e\} - \{a,e\} + \\ &\qquad \{a,c\} - \{b,c\} - \{a,b\} + \{a,d\} + \{d,e\} - \{a,e\}] \\ &= 2\cdot[\{a,d\} + \{d,e\} - \{a,e\}] = 2\cdot\gamma_1. \end{aligned}$$

Hence ∂_2 corresponds to the map $\mathbb{Z} \to \mathbb{Z}^2$ which sends k to $(2k, 0)$. This yields

$$H_1(K) \cong \operatorname{coker}(\partial_1) = {(\mathbb{Z}\cdot\gamma_1 \oplus \mathbb{Z}\cdot\gamma_2)}\big/{\mathbb{Z}\cdot(2\gamma_1, 0)} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}.$$

[4]More precisely, one can assign signs to 2-simplices of K_2 in a way what the cycle given by the sum of these has $(\{f,g\} + \{g,h\} + \{h,i\} - \{f,i\} - (\{a,b\} + \{b,c\} - \{a,c\} + \{a,d\} + \{d,e\} - \{a,e\} + \{a,c\} - \{b,c\} - \{a,b\} + \{a,d\} + \{d,e\} - \{a,e\})$ as its boundary, but we refrain from doing the long calculation here.

Moreover, we have $0 = \ker \partial_1 = \operatorname{im} \rho_2$. Hence $\ker \rho_2 = H_2(K)$, but $\ker \rho_2 = \operatorname{im} \iota_2 = 0$, so $H_2(K) \cong 0$.

All in all, we have:

$$H_n(K) \cong \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & \text{otherwise} \end{cases} .$$