On descendable algebras

aka some results from A. Mathew's "The galois group of a stable homotopy theory"

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Outline

Convention

Fix a (presentable) stable ∞-category $\mathcal{C}$ with a symmetric monoidal structure (with tensor product $\otimes$ and unit $\mathbf{1}$) such that the tensor product commutes with colimits (or as Mathew calls it, a "stable homotopy theory").

Some tensor triangular algebra

Definition

A full subcategory $\mathcal{D} \subseteq \mathcal{C}$ is called *thick* of if it is closed under finite limits, finite colimits and retracts.

Definition

A full subcategory $\mathcal{I} \subseteq \mathcal{C}$ is called a *$\otimes$-ideal* if for all $A \in \mathcal{C}$, $X \in \mathcal{I}$, $A \otimes X \in \mathcal{I}$.

Definition

A *thick $\otimes$-ideal* is a thick subcategory that is also a $\otimes$-ideal.

Descendable algebras

Definition

A commutative algebra $A \in \mathrm{Alg}_{\mathrm{Comm}}(\mathcal{C})$ is called *descendable* if $\mathbf{1}$ is in the thick $\otimes$-ideal generated by A.

Remark

This also yields a concept of descendable morphism by considering a morphism $f : A \to B$ *of commutative algebras as an object in* $\mathrm{Alg}_{\mathrm{Comm}}(\mathrm{Mod}_{\mathcal{C}}(A))$ *(and* $\mathrm{Mod}_{\mathcal{C}}(A)$ *as the "stable homotopy theory" in the background).*

Examples of descendable algebras

Example

Let R be a commutative ring spectrum. Then the following are descendable R-algebras:

1. If R is discrete and $I \subseteq R$ is a nilpotent ideal, then R/I.
2. $R[x^{-1}] \times R_x^{\wedge}$ for $x \in \pi_0 R$.
3. Any finite faithful Galois extension of R.
4. Any R-algebra A such that
   - $\pi_0 R \to \pi_0 A$ is faithfully flat,
   - for $i > 0$, $\pi_i R \otimes_{\pi_0 R} \pi_0 A \to \pi_i A$ is an isomorphism and
   - $\pi_0 A$ has a presentation as a $\pi_0 R$-algebra with at most $\aleph_k$ generators and relations for some $k \in \mathbb{N}$.
5. If R is connective and $\pi_i R \cong 0$ for large enough i, then $\pi_0 R$.

The category of pro-objects

Definition

The *category of pro-objects in* $\mathcal{C}$ is

$$\mathsf{Pro}(\mathcal{C}) := \mathsf{Ind}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}.$$

Proposition (Modulo size issues; HTT 5.3.5.10.)

There is a functor $\iota\colon \mathcal{C} \to \mathsf{Pro}(\mathcal{C})$ *such that for every* ∞*-category* $\mathcal{D}$ *admitting cofiltered limits, restriction along* ι *induces an equivalence*

$$\mathsf{Fun}^{\textit{cofilt-cont}}(\mathsf{Pro}(\mathcal{C}), \mathcal{D}) \simeq \mathsf{Fun}(\mathcal{C}, \mathcal{D})$$

where the left hand side denotes the category of functors that commute with cofiltered limits.

Diagrams as pro-objects

Convention

Given a cofiltered diagram $F\colon I \to \mathcal{C}$, we will consider it as an object of $\mathrm{Pro}(\mathcal{C})$ by taking the limit of the composite

$$I \xrightarrow{F} \mathcal{C} \xrightarrow{\iota} \mathrm{Pro}(\mathcal{C}).$$

Constant pro-objects

Definition

A pro-object is called *constant* if it is in the essential image of $\iota \colon \mathcal{C} \to \mathrm{Pro}(\mathcal{C})$.

Example

The pro-object associated to a constant diagram is a constant pro-object.

A criterion for being constant

Proposition

The pro-object associated to a cofiltered diagram $F\colon I \to \mathcal{C}$ is constant if and only if F admits a limit in $\mathcal{C}$ that is preserved by every functor that preserves finite limits.

Example

If $X^\bullet$ is a cosimplicial object that admits a split coaugmentation, then the associated tower

$$\ldots \to \operatorname{Tot}_2 X^\bullet \to \operatorname{Tot}_1 X^\bullet \to \operatorname{Tot}_0 X^\bullet$$

of partial totalizations defines a constant pro-object.

The Amitsur complex of a descendable algebra – the statement

Proposition

$A \in \mathrm{Alg}_{\mathrm{Comm}}\,\mathcal{C}$ is descendable if and only if the map $\mathrm{const}_{\mathbf{1}} \to C^\bullet(A)$ induced by the cougmentation of the Amitsur complex induces an equivalence between the pro-objects associated to the respective towers of partial totalizations.

Remark

The condition in the proposition means in particular that the tower of partial totalizations associated to $C^\bullet(A)$ is pro-constant.

Corollary

(Exact?) (strong?) symmetric monoidal fuctors preserve descendable algebras.

The Amitsur complex of a descendable algebra – a proof sketch

Proof sketch.

($\implies$) Check that the subcategory spanned all $X \in \mathcal{C}$ such that $\mathrm{const}_X \to C^\bullet(A) \otimes X$ induces a pro-equivalence between the associated towers of partial totalizations is a thick $\otimes$-ideal. Note that A is in this subcategory because $C_+^\bullet(A) \otimes A$ is split, so, by descendability, $\mathbf{1}$ is too.

($\impliedby$): The (homotopy) inverse of the map induced by $\mathrm{const}_{\mathbf{1}} \to C^\bullet(A)$ between the pro-objects associated to towers of partial totalizations amounts to a map $\mathrm{Tot}_n(C^\bullet(A)) \to \mathbf{1}$ for large enough n such that the composite $\mathbf{1} \to \mathrm{Tot}_n(C^\bullet(A)) \to \mathbf{1}$ is homotopic to the identity. Now $\mathrm{Tot}_n(C^\bullet(A))$ is in the thick $\otimes$-ideal generated by A, so $\mathbf{1}$, which is a retract of the former, is too. □

Descendability and comonadicity

Proposition

If A is a descendable algebra, then the extension-restriction of scalars adjunction $\mathcal{C} \rightleftarrows \mathrm{Mod}_{\mathcal{C}}(A)$ is comonadic.

Proof sketch.

We use the Barr–Beck–Lurie (co)monadicity theorem:

1. Let $M \in \mathcal{C}$ such that $M \otimes A \simeq 0$. Then the subcategory spanned by $X \in \mathcal{C}$ such that $M \otimes X \simeq 0$ is a thick $\otimes$-ideal containing A. Hence it contains $\mathbf{1}$, which implies $M \simeq M \otimes \mathbf{1} \simeq 0$. Thus $(-) \otimes A$ is conservative.
2. Let $X^{\otimes}$ be an A-split cosimplicial object in $\mathcal{C}$. Check that the subcategory spanned by $Y \in \mathcal{C}$ such that $X^{\bullet} \otimes Y$ has a pro-constant tower is a thick $\otimes$-ideal containing A. Hence it contains $\mathbf{1}$, which implies that $X^{\bullet}$ admits a limit which is preserved under tensoring with A.

□