The localization of spectra with respect to homology

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Outline

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Conventions

The labeling of statements refers to the numbering in Bousfield's paper.

Warning

What the category Sp of spectra is is intentionally kept vague.
Depending on whether Sp is the stable homotopy category, a point-set model or the ∞-category of spectra, the statements may mean slightly different things and may be stronger or weaker.
Bousfield uses the stable homotopy category and CW-spectra.

Assumption

The smash product $\wedge : \mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$ is assumed to be "homotopically correct", in particular exact in both variables.

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E-equivalences

Definition

A map $f \colon X \to Y$ of spectra is called an ***E*-equivalence** if $E_* f \colon E_* X \to E_* Y$ is an isomorphism.

We would like to have have a category Sp_E equipped with a “localization functor” $(-)_E \colon \mathrm{Sp} \to \mathrm{Sp}_E$ s.t.

$$f \colon X \to Y \text{ is an } E\text{-equivalence} \iff f_E \colon X_E \to Y_E \text{ is an equivalence.}$$

Spoiler

In the end, we will be able to realize the target of the localization functor as the full subcategory of Sp consisting of “E-local” spectra.

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Why should we care about E-equivalences? (I)

Example (Proposition 2.9)

Each spectrum X sits in a homotopy pullback square

$$\begin{array}{ccc} X & \longrightarrow & \prod_{p \text{ prime}} X_{\mathbb{S}/p} \\ \downarrow & & \downarrow \\ X_{\mathbb{SQ}} & \longrightarrow & \left(\prod_{p \text{ prime}} X_{\mathbb{S}/p}\right)_{\mathbb{SQ}} \end{array},$$

a.k.a. an "arithmetic square".

Why should we care about E-equivalences? (II)

Example (a consequence of Theorem 6.6)

Let E be a connective ring spectrum such that $\pi_0 E \cong \mathbb{Z}/n$ for some $n \geqslant 2$.
Let Y be a connective spectrum with finitely generated homotopy groups.
Then the E-based Adams spectral sequence for Y converges to $\pi_* Y_{\mathbb{S}/n}$.

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E-acyclicity and E-locality

Definition

A spectrum X is called

- E-**acyclic** if $E_* X \cong 0$, i.e. $E \wedge X \simeq 0$.
- E-**local** if for each E-equivalence $f \colon A \to B$, $f^* \colon [B, X]_\bullet \to [A, X]_\bullet$ is a bijection.

Lemma

A map $f \colon X \to Y$ is an E-equivalence if and only if its homotopy (co)fiber is E-acyclic.

Corollary

A spectrum X is E-local iff for every E-acyclic spectrum A, $[A, X]_\bullet \cong 0$.

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A class of E-local spectra

Lemma (Lemma 1.3)

If E is a ring spectrum (up to homotopy), then all E-module spectra are E-local.

Proof.

Let A be an E-acyclic spectrum, $f\colon A \to X$.
Then, up to homotopy, f can be factored as

$$A \xrightarrow{1_E \wedge A} E \wedge A \xrightarrow{E \wedge f} E \wedge X \xrightarrow{\mathrm{act}_X} X.$$

Since A is E-acyclic, $E \wedge A \simeq 0$.
Thus f factors through 0, so $f \sim 0$. □

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E-equivalences between E-local spectra

Fact (Lemma 1.2, "E-Whitehead theorem")

Let $f : X \to Y$ be an E-equivalence between E-local spectra. Then f is an equivalence.

Closure properties of E-local spectra

Fact (Lemmas 1.4-1.8)

The subcategory Sp_E of E-local spectra is closed under

- *homotopy (co)fibers,*
- *homotopy limits,*
- *extensions,*
- *retracts.*

Remark

The dual statements hold for E-acyclic spectra.

Warning

In general, Sp_E is not closed under smash products.

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The existence theorem

Theorem (Theorem 1.1)

There are functors

- ${}_E(-)\colon \mathrm{Sp} \to \mathrm{Sp}$ (*E*-**acyclization**) *which lands in E-acyclic spectra,*
- $(-)_E\colon \mathrm{Sp} \to \mathrm{Sp}$ (*E*-**localization**) *which lands in E-local spectra,*

such that for each spectrum X there exists a natural homotopy (co)fiber sequence

$$ {}_E X \xrightarrow{\theta_X} X \xrightarrow{\eta_X} X_E. $$

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$\eta_E \colon X \to X_E$ is an E-equivalence.

Have: hocofiber sequence ${}_EX \xrightarrow{\theta_X} X \xrightarrow{\eta_X} X_E$ s.t. ${}_EX$ is E-acyclic and X_E is E-local.

Corollary

$\eta_X \colon X \to X_E$ is an E-equivalence.

Proof.
Smashing the localization sequence with E yields a homotopy (co)fiber sequence

$$0 \simeq E \wedge {}_EX \xrightarrow{E \wedge \theta_X} E \wedge X \xrightarrow{E \wedge \eta_X} E \wedge X_E,$$

so $E_*\eta_X = \pi_*(E \wedge \eta_X)$ is an equivalence. □

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Idempotency of the localization functor

Have: hocofiber sequence ${}_E X \xrightarrow{\theta_X} X \xrightarrow{\eta_X} X_E$ s.t. ${}_E X$ is E-acyclic and X_E is E-local.

Corollary

The functor $(-)_E \colon \mathrm{Sp} \to \mathrm{Sp}$ is idempotent (up to homotopy).

Proof.
$\eta_{X_E} \colon X_E \to (X_E)_E$ is an E-equivalence between E-local spectra.
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Corollary

$\eta_X \colon X \to X_E$ is (up to homotopy) initial among maps from X to an E-local spectrum.

Proof.

If Y is E-local, then $\eta_X^* \colon [X, Y] \cong [X_E, Y]$ since η_E is an E-equivalence. □

Corollary

E-localization is left adjoint to the inclusion $\mathrm{Sp}_E \hookrightarrow \mathrm{Sp}$ of E-local spectra.

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Corollary

$\theta_X\colon {}_EX \to X$ is (up to homotopy) terminal among maps from an E-acyclic spectrum to X.

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E-acyclization is right adjoint to the inclusion $\mathrm{Sp}_E \hookrightarrow \mathrm{Sp}$ of E-acyclic spectra.

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Exactness of acyclization and localization functors

Corollary

E-acyclization and E-localization are exact functors.

Outline

How to construct localizations?

Recipe for constructing X_E.

1. Construct a spectrum aE such that $[A, Y]_\bullet \cong 0$ for all E-acyclic A iff $[aE, Y]_\bullet \cong 0$.
2. "Kill" all the maps from aE to X.

We'll sketch these constructions for CW-spectra (i.e. sequential spectra $(X_n)_{n\in\mathbb{N}}$ s.t. every level X_n is a CW-complex and the structure maps $\Sigma X_n \to \Sigma X_{n+1}$ are inclusions of subcomplexes).

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“The” acyclic spectrum

Fix an infinite cardinal σ that is at least equal to $|\bigoplus_{n\in\mathbb{Z}} \pi_n E|$.

Definition

Let $(K_i)_{i\in I}$ a system of representatives for the equivalence classes of E-acyclic spectra with at most σ cells.

Set

$$aE := \bigvee_{i\in I} K_i.$$

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$$aE := \bigvee_{i\in I} K_i.$$

"The" acyclic spectrum

Fix an infinite cardinal σ that is at least equal to $|\bigoplus_{n\in\mathbb{Z}} \pi_n E|$.

Definition

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aE "generates" all E-acyclic spectra (I)

Let Y be a spectrum s.t. $[aE, Y]_\bullet \cong 0$.

We want to show that $[A, Y]_\bullet \cong 0$ for **every** E-acyclic spectrum A.

To do that, we would like to construct a (transfinite) filtration

$$0 = A_0 \subset A_1 \subset \ldots \subset A_\alpha = A$$

by E-acyclic CW-subspectra s.t.

1. $A_{\gamma+1} = A_\gamma \cup W_\gamma$ for a subspectrum $W_\gamma \subset A$ s.t.
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Note that the successor step guarantees that the subquotients $A_{\gamma+1}/A_\gamma$ are E-acyclic spectra with at most σ cells, so they are all "summands" of $aE = \bigvee_i K_i$. Thus, by (transfinite) induction along this filtration, we can show that $[A, Y]_\bullet \cong 0$ if $[aE, Y]_\bullet \cong 0$.

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How to do the successor step?

Lemma

Let A be a CW-spectrum.
Let $B \subset A$ a proper closed subspectrum with $E_*(A/B) \cong 0$.
Let e be a cell of A that is not in B.
Then there exists a CW-subspectrum $W \subset A$ such that:

- *W contains e.*
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and set $W := \bigcup_n W_n$.

Let W_0 be a CW-subspectrum of A with at most σ cells that contains e.

Given W_n, consider $x \in E_*(W_n/(W_n \cap B))$. As $E_*(A/B) \cong 0$, there exists a finite CW-subspectrum $F_x \subset X$ s.t. x maps to 0 in $E_*((W_n \cup F_x)/((W_n \cup F_x) \cap B))$.

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Recap of the construction of X_E

We have constructed a spectrum aE such that $[aE, Y]_\bullet \cong 0$ iff $[A, Y]_\bullet \cong 0$ for every E-acyclic spectrum A.

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The small object argument

Given a (CW-)spectrum X, construct X_E by (transfinite) induction as follows:

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$$\bigvee_{n \in \mathbb{Z}} \bigvee_{[f] \in [aE, X_\alpha]_n} S^i \xrightarrow{\bigvee_n \bigvee_{[f]} f} X_\alpha.$$

- At limit ordinals λ set $X_\lambda := \operatorname{hocolim}_{i<\lambda} X_i$.

Pick a cardinal κ larger than the number of cells in aE. Set $X_E := X_\kappa$. This guarantees that every map $\Sigma^i aE \to X_E$ factors through X_i for some $i < \kappa$, so is trivial because it gets coned off at the next stage.

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Pick a cardinal κ larger than the number of cells in aE. Set $X_E := X_\kappa$.
This guarantees that every map $\Sigma^i aE \to X_E$ factors through X_i for some $i < \kappa$, so is trivial because it gets coned off at the next stage.

The small object argument

Given a (CW-)spectrum X, construct X_E by (transfinite) induction as follows:

- Let $X_0 := X$.
- Given X_α, define $X_{\alpha+1}$ to be the (homotopy) cofiber of

$$\bigvee_{n\in\mathbb{Z}} \bigvee_{[f]\in[aE,X_\alpha]_n} S^i \xrightarrow{\bigvee_n \bigvee_{[f]} f} X_\alpha.$$

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Outline

Bousfield classes

Let F be another spectrum.

Definition

E and F are called **Bousfield equivalent** if one of the following equivalent conditions holds:

1. A spectrum is E-acyclic iff it is F-acyclic.
2. A map between spectra is an E_*-equivalence iff it is an F_*-equivalence.

The equivalence class of E w.r.t. this relation will be called the **Bousfield class of E** and denoted by $\langle E \rangle$.

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The set of Bousfield classes as a lattice

Fact

The set (!) of Bousfield classes of spectra is a lattice with

- *join induced by wedge of spectra,*
- *meet induced by smash product of spectra.*

In particular, $\langle 0 \rangle$ is the minimal element and $\langle \mathbb{S} \rangle$ is the maximal element.

Definition

We define a partial order on the set of Bousfield clasess by declaring $\langle E \rangle \leqslant \langle F \rangle$ if every F-acyclic spectrum is E-acyclic.

Remark

This order agrees with the one coming from the lattice structure.

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Acyclicity types of abelian groups

Definition

Two abelian groups G_1 and G_2 have the **same type of acyclicity** if

- G_1 is a torsion group iff G_2 is, and
- for each prime p, G_1 is uniquely p-divisible iff G_2 is.

Fact (Proposition 2.3)

For abelian groups G_1 and G_2, the following are equivalent:

1. *G_1 and G_2 have the same type of acyclicity.*
2. $\langle \mathbb{S}G_1 \rangle = \langle \mathbb{S}G_2 \rangle$.
3. *$\mathbb{S}G_1$ and $\mathbb{S}G_2$ yield equivalent localization functors on* Sp.

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An explicit description of acyclicity types

Remark

Every acyclicity class is represented by one of the following:

- $\prod_{p \in J} \mathbb{Z}/p$ for a set J of primes,
- $\mathbb{Z}_{(J)}$ for a set J of primes.

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The **complement** of an acyclicity type (or by abuse of terminology, an abelian group) is defined as follows:

- If $\prod_{p \in J} \mathbb{Z}/p$ is in the class for a set J of primes, then the complement contains $\mathbb{Z}_{(J)}$.
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Example

(The acyclicity classes of) $\mathbb{Q}$ and $\prod_{p \text{ prime}} \mathbb{Z}/p$ are complements of each other.

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Remark

Let G be an abelian group and G' an abelian group in the complement of its acyclicity type.

Then:

- $G \oplus G'$ and $\mathbb{Z}$ have the same type of acyclicity.
- $\langle \mathbb{S}G \vee \mathbb{S}G' \rangle = \langle \mathbb{S} \rangle$.

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Outline

The (generalized) arithmetic square

Theorem (Proposition 2.9)

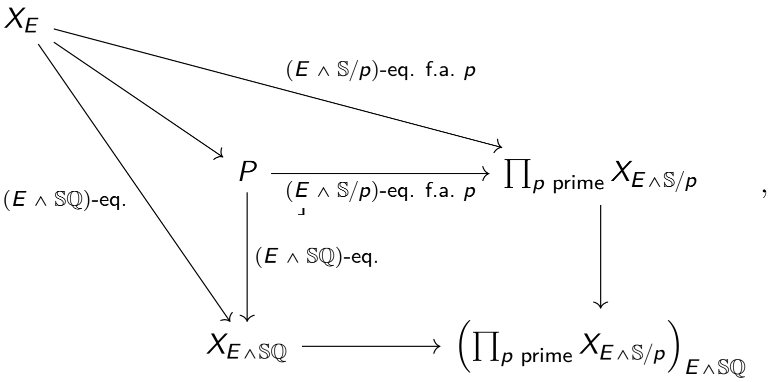
Each spectrum X sits in a homotopy pullback square

$$\begin{array}{ccc} X_E & \longrightarrow & \prod_{p \text{ prime}} X_{E \wedge \mathbb{S}/p} \\ \downarrow & & \downarrow \\ X_{E \wedge \mathbb{SQ}} & \longrightarrow & \left(\prod_{p \text{ prime}} X_{E \wedge \mathbb{S}/p}\right)_{E \wedge \mathbb{SQ}} \end{array},$$

where all the maps are induced by corresponding localizations.

Proof of the arithmetic square theorem

Proof sketch.



The homotopy pullback P is E-local as a limit of E-local spectra, so it's enough to show that $X_E \to P$ is an E-equivalence. □

Localizations of connective spectra w.r.t. connective spectra

Theorem (Theorem 3.1)

Assume that E is connective.
Let X be a connective spectrum.
Then $X_E \simeq X_{\mathbb{S}(\bigoplus_{n\in\mathbb{Z}}\pi_n E)}$.

Corollary

Let G be an abelian group, X a connective spectrum.
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A “telescope theorem”

Theorem (Proposition 4.2)

Let p be a prime number.

Let $A_p\colon \Sigma^{2(p-1)}\mathbb{S}/p \to \mathbb{S}/p$ for p odd resp. $A_p\colon \Sigma^8\mathbb{S}/2 \to \mathbb{S}/2$ for $p = 2$ be the Adams map.

Then the natural map

$$\mathbb{S}/p \to \operatorname{hocolim}(\mathbb{S}/p \xrightarrow{\Sigma^{-\deg A_p} A_p} \Sigma^{-\deg A_p}\mathbb{S}/p \xrightarrow{\Sigma^{-2\deg A_p} A_p} \dots)$$

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